Script generated by TTT

Title: Seidl: Programmoptimierung (04.11.2013)

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Pages: 57

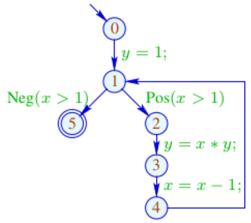
Conclusion:

Systems of inequations can be solved through fixpoint iteration, i.e., by repeated evaluation of right-hand sides :-)

Caveat:

Naive fixpoint iteration is rather inefficient :-(

Example:



	1	2	3	4	5
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	
1	$\{1, x > 1, x - 1\}$	$\{1\}$	$\{1\}$	$\{1\}$	
2	$Expr$	$\{1, x > 1, x - 1\}$	$\{1, x > 1\}$	$\{1, x > 1\}$	
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Instead of accessing the values of the last iteration, always use the current values of unknowns :-)

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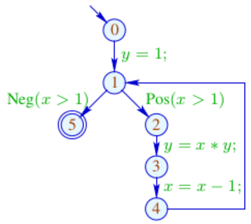

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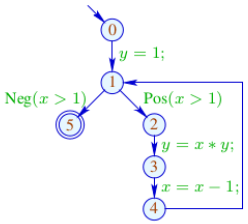

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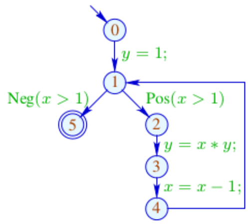

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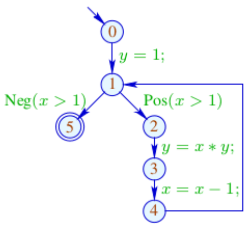

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The code for Round Robin Iteration in Java looks as follows:

```
for (i = 1; i ≤ n; i++) x_i = ⊥;
do {
    finished = true;
    for (i = 1; i ≤ n; i++) {
        new = f_i(x_1, ..., x_n);
        if (!(x_i ⊒ new)) {
            finished = false;
            x_i = x_i ⊔ new;
        }
    }
} while (!finished);
```

$x_i \sqsupseteq f(\dots)$

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Assume $y_i^{(d)}$ is the i-th component of $F^d \perp$.

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Good:

→ u before v, if $u \to^* v$;

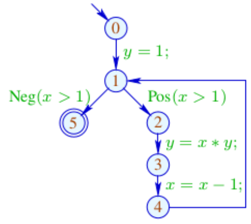
→ entry condition before loop body :-)

Bad:

e.g., post-order DFS of the CFG, starting at start :-)

$u \longrightarrow v$

Good:



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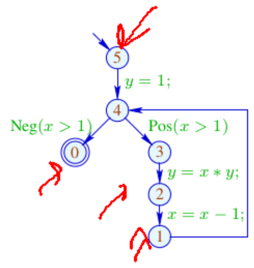
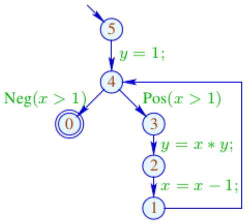


Inefficient Round Robin Iteration:



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$\Longrightarrow$ significantly less efficient :-)

... end of background on: Complete Lattices

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Final Question:

Why is a (or the least) solution of the constraint system useful ???

$$\mathcal{A}[v] = \bigcap \{ [\![\pi]\!]^\# \phi \mid \pi : start \to v \}$$

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Why is a (or the least) solution of the constraint system useful ???

For a complete lattice $\mathbb{D}$, consider systems:

$$\begin{array}{lll} \mathcal{I}[start] & \sqsupseteq & d_0 \\ \mathcal{I}[v] & \sqsupseteq & [\![k]\!]^\sharp (\mathcal{I}[u]) \qquad k = (u, \_, v) \quad \text{edge} \end{array}$$

where $d_0 \in \mathbb{D}$ and all $[\![k]\!]^\sharp : \mathbb{D} \to \mathbb{D}$ are monotonic ...

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$\Longrightarrow$ Monotonic Analysis Framework

Wanted: MOP (Merge Over all Paths)

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Theorem Kam, Ullman 1975

Assume $\mathcal{I}$ is a solution of the constraint system. Then:

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Jeffrey D. Ullman, Stanford

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In particular: $\mathcal{I}[v] \sqsupseteq [\![\pi]\!]^\sharp\, d_0$ for every $\pi : start \to^* v$

Proof: Induction on the length of π.

Foundation: $\pi = \epsilon$ (empty path)

Then:

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Step: $\pi = \pi' k$ for $k = (u, \_, v)$ edge.

Then:

$$\begin{array}{rlll} & [\![\pi']\!]^\sharp\, d_0 & \sqsubseteq & \mathcal{I}[u] & \text{by I.H. for } \pi \\ \Longrightarrow & [\![\pi]\!]^\sharp\, d_0 & = & [\![k]\!]^\sharp\,([\![\pi']\!]^\sharp\, d_0) \\ & & \sqsubseteq & [\![k]\!]^\sharp\,(\mathcal{I}[u]) & \text{since } [\![k]\!]^\sharp \text{ monotonic} \\ & & \sqsubseteq & \mathcal{I}[v] & \text{since } \mathcal{I} \text{ solution} \quad \text{:-))} \end{array}$$

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Are solutions of the constraint system just upper bounds ???

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With the notable exception when all functions $[\![k]\!]^\sharp$ are distributive ... :-)

The function $f : \mathbb{D}_1 \to \mathbb{D}_2$ is called

- distributive, if $f\,(\bigsqcup X) = \bigsqcup\{f\,x \mid x \in X\}$ for all $\emptyset \neq X \subseteq \mathbb{D}$;
- strict, if $f\,\bot = \bot$.
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$\bigsqcup \emptyset = \bot$

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$\mathbb{D} = 2^{U}$

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  Strictness: $f\,\emptyset = a \cap \emptyset \cup b = b = \emptyset$ whenever $b = \emptyset$:-(

  Distributivity:

$$
\begin{aligned}
f\,(x_1 \cup x_2) &= a \cap (x_1 \cup x_2) \cup b \\
&= a \cap x_1 \cup a \cap x_2 \cup b \\
&= f\,x_1 \cup f\,x_2 \qquad :\text{-)}
\end{aligned}
$$

$\cup\, b$

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  Distributivity:

$$\begin{array}{rcl} f\,((1,4) \sqcup (4,1)) & = & f\,(4,4) \quad = \quad 8 \\ \neq \quad 5 & = & f\,(1,4) \sqcup f\,(4,1) \qquad :\text{-)} \end{array}$$

Remark:

If $f : \mathbb{D}_1 \to \mathbb{D}_2$ is distributive, then also monotonic :-)

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From that follows:

$$\begin{array}{rrcl} & f\,b & = & f\,(a \sqcup b) \\ & & = & f\,a \sqcup f\,b \\ \Longrightarrow & f\,a & \sqsubseteq & f\,b \end{array} \qquad \text{:-)}$$

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Then:

Theorem Kildall 1972

If all effects of edges $[\![k]\!]^\sharp$ are distributive, then: $\mathcal{I}^*[v] = \mathcal{I}[v]$
for all v .

Gary A. Kildall (1942-1994).

Has developed the operating system CP/M and GUIs for PCs.

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$\subseteq$

$\supseteq$

Proof:

It suffices to prove that $\mathcal{I}^*$ is a solution :-)

For this, we show that $\mathcal{I}^*$ satisfies all constraints :-))

(1) We prove for *start* :

$$\begin{array}{lll} \mathcal{I}^*[start] & = & \bigsqcup\{[\![\pi]\!]^\sharp\, d_0 \mid \pi : start \to^* start\} \\ & \sqsupseteq & [\![\epsilon]\!]^\sharp\, d_0 \\ & \sqsupseteq & d_0 \qquad \text{:-)} \end{array}$$

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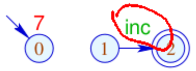
(2) For every $k = (u, \_, v)$ we prove:

$$\begin{array}{lll} \mathcal{I}^*[v] & = & \bigsqcup\{[\![\pi]\!]^\sharp\, d_0 \mid \pi : start \to^* v\} \\ & \sqsupseteq & \bigsqcup\{[\![\pi' k]\!]^\sharp\, d_0 \mid \pi' : start \to^* u\} \\ & = & \bigsqcup\{[\![k]\!]^\sharp\, ([\![\pi']\!]^\sharp\, d_0) \mid \pi' : start \to^* u\} \\ & = & [\![k]\!]^\sharp\, (\bigsqcup\{[\![\pi']\!]^\sharp\, d_0 \mid \pi' : start \to^* u\}) \\ & = & [\![k]\!]^\sharp\, (\mathcal{I}^*[u]) \end{array}$$

since $\{\pi' \mid \pi' : start \to^* u\}$ is non-empty :-)

Caveat:

- Reachability of all program points cannot be abandoned! Consider:



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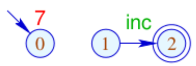

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Then:

$$\begin{array}{lllll} \mathcal{I}[2] & = & \mathsf{inc}\, 0 & = & 1 \\ \mathcal{I}^*[2] & = & \bigsqcup \emptyset & = & 0 \end{array}$$

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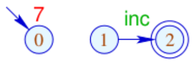

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- Unreachable program points can always be thrown away :-)

Summary and Application:

→ The effects of edges of the analysis of availability of expressions are distributive:

$$\begin{array}{lll} (a \cup (x_1 \cap x_2))\backslash b & = & ((a \cup x_1) \cap (a \cup x_2))\backslash b \\ & = & ((a \cup x_1)\backslash b) \cap ((a \cup x_2)\backslash b) \end{array}$$

$$X \backslash b = X \cap \overline{b}$$