Script **generated by TTT**

Title: Seidl: Programmoptimierung (23.10.2013)

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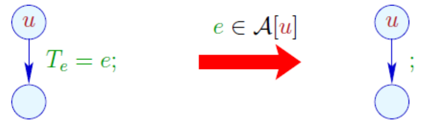
Duration: 85:44 min

Pages: 42

... analogously for $R = M[e];$ and $M[e_1] = e_2;$.

Transformation 1.2:

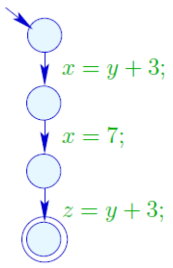
If e is available at program point u, then e need not be re-evaluated:



We replace the assignment with Nop :-)

Example:

$$
\begin{array}{lcl}
x & = & \boxed{y+3;} \\
x & = & 7; \\
z & = & \boxed{y+3;}
\end{array}
$$



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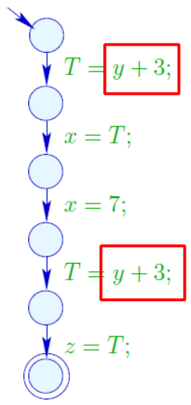

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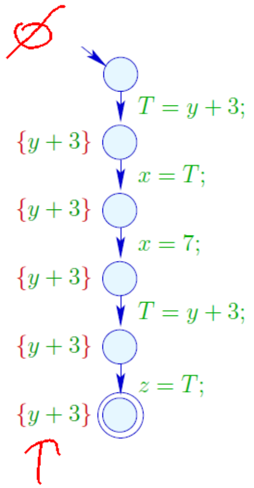

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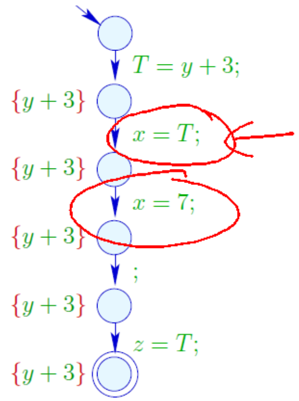


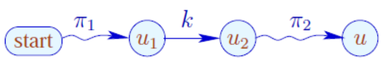
Correctness: (Idea)

Transformation 1.1 preserves the semantics and $\mathcal{A}[u]$ for all program points u :-)

Assume $\pi : start \to^* u$ is the path taken by a computation.

If $e \in \mathcal{A}[u]$, then also $e \in [\![\pi]\!]^\sharp \emptyset$.

Therefore, π can be decomposed into:



with the following properties:

- The expression e is evaluated at the edge k;
- The expression e is not removed from the set of available expressions at any edge in π_2, i.e., no variable of e receives a new value :-)

$$\Longrightarrow$$

The register T_e contains the value of e whenever u is reached :-))

Warning:

Transformation 1.1 is only meaningful for assignments $x = e;$ where:

→ $x \notin Vars(e)$;

→ $e \notin Vars$;

→ the evaluation of e is non-trivial :-}

Which leaves us with the following question ...

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How do we compute $\mathcal{A}[u]$ for every program point u ??

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We collect all restrictions to the values of $\mathcal{A}[u]$ into a system of constraints:

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$$A[v] = \bigcap \{ [\![k]\!]^\sharp A[u] \mid k = (u, \_, v) \}$$

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Wanted:

- a maximally large solution (??)
- an algorithm which computes this :-)

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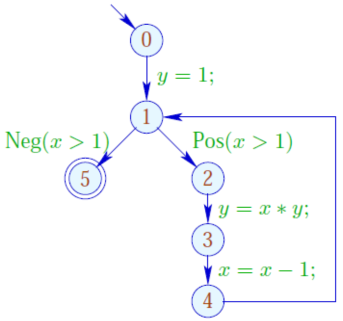

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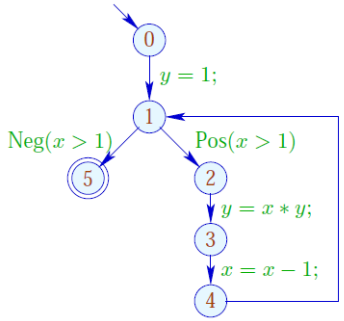

$$\mathcal{A}[0] \subseteq \emptyset$$

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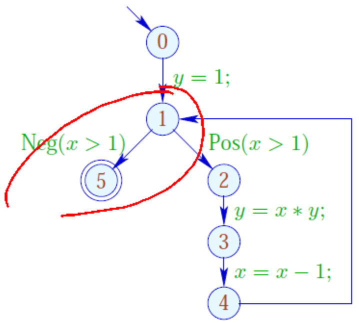

$$\begin{array}{lcl}
\mathcal{A}[0] & \subseteq & \emptyset \\
\mathcal{A}[1] & \subseteq & (\mathcal{A}[0] \cup \{1\}) \backslash Expr_y \\
\mathcal{A}[1] & \subseteq & \mathcal{A}[4] \\
\mathcal{A}[2] & \subseteq & \mathcal{A}[1] \cup \{x > 1\} \\
\mathcal{A}[3] & \subseteq & (\mathcal{A}[2] \cup \{x * y\}) \backslash Expr_y \\
\mathcal{A}[4] & \subseteq & (\mathcal{A}[3] \cup \{x - 1\}) \backslash Expr_x \\
\mathcal{A}[5] & \subseteq & \mathcal{A}[1] \cup \{x > 1\}
\end{array}$$

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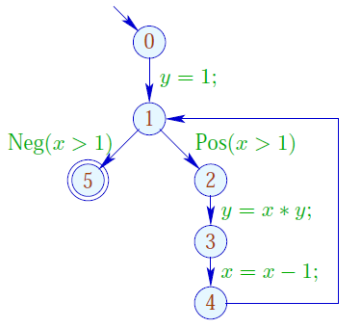

Solution:

$$\begin{array}{lcl}
\mathcal{A}[0] & = & \emptyset \\
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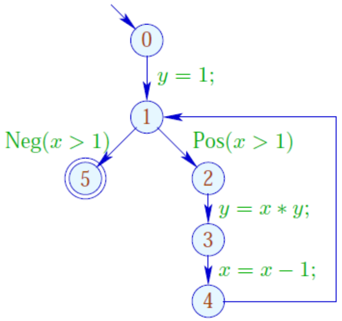

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Observation:

- The possible values for $\mathcal{A}[u]$ form a complete lattice:

$$\mathbb{D} = 2^{Expr} \quad \text{with} \quad B_1 \sqsubseteq B_2 \quad \text{iff} \quad B_1 \supseteq B_2$$

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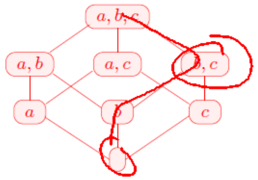
Background 2: Complete Lattices

A set $\mathbb{D}$ together with a relation $\sqsubseteq \subseteq \mathbb{D} \times \mathbb{D}$ is a partial order if for all $a, b, c \in \mathbb{D}$,

$a \sqsubseteq a$	*reflexivity*
$a \sqsubseteq b \wedge b \sqsubseteq a \implies a = b$	*anti−symmetry*
$a \sqsubseteq b \wedge b \sqsubseteq c \implies a \sqsubseteq c$	*transitivity*

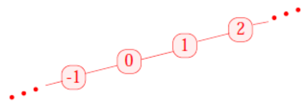
Examples:

1. $\mathbb{D} = 2^{\{a,b,c\}}$ with the relation "$\subseteq$" :

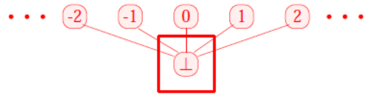


2. $\mathbb{Z}$ with the relation "$=$" :

3. $\mathbb{Z}$ with the relation "$\leq$" :



4. $\mathbb{Z}_\perp = \mathbb{Z} \cup \{\perp\}$ with the ordering:



$d \in \mathbb{D}$ is called upper bound for $X \subseteq \mathbb{D}$ if

$$x \sqsubseteq d \qquad \text{for all } x \in X$$

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Caveat:

- $\{0, 2, 4, \ldots\} \subseteq \mathbb{Z}$ has no upper bound!
- $\{0, 2, 4\} \subseteq \mathbb{Z}$ has the upper bounds $4, 5, 6, \ldots$

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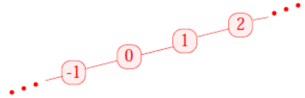

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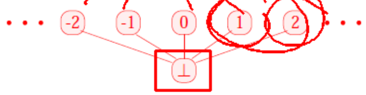


A complete lattice (cl) $\mathbb{D}$ is a partial ordering where every subset $X \subseteq \mathbb{D}$ has a least upper bound $\bigsqcup X \in \mathbb{D}$.

Note:

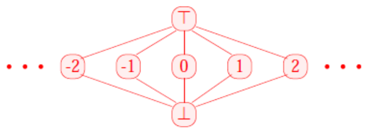
Every complete lattice has

→ a least element $\perp = \bigsqcup \emptyset \in \mathbb{D}$;

→ a greatest element $\top = \bigsqcup \mathbb{D} \in \mathbb{D}$.

Examples:

1. $\mathbb{D} = 2^{\{a,b,c\}}$ is a cl :-)
2. $\mathbb{D} = \mathbb{Z}$ with "$=$" is not.
3. $\mathbb{D} = \mathbb{Z}$ with "$\leq$" is neither.
4. $\mathbb{D} = \mathbb{Z}_\perp$ is also not :-(
5. With an extra element $\top$, we obtain the flat lattice $\mathbb{Z}_\perp^\top = \mathbb{Z} \cup \{\perp, \top\}$:



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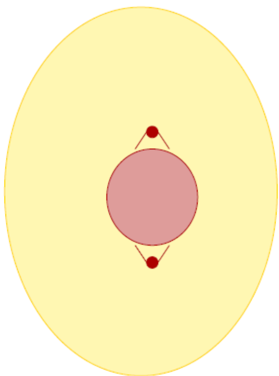
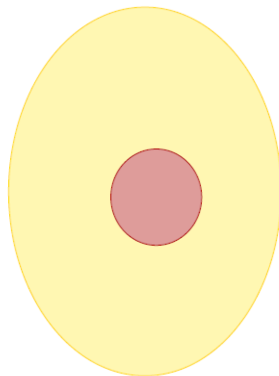
Proof:

Construct $U = \{u \in \mathbb{D} \mid \forall x \in X : u \sqsubseteq x\}$.

// the set of all lower bounds of X :-)





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Set: $g := \bigsqcup U$

Claim: $g = \bigsqcap X$

(1) g is a lower bound of X :

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(2) g is the greatest lower bound of X :

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Because:

$$x \sqsupseteq d_1 \wedge \ldots \wedge x \sqsupseteq d_k \quad \text{iff} \quad x \sqsupseteq \bigsqcup\{d_1, \ldots, d_k\} \qquad :-)$$

A mapping $f : \mathbb{D}_1 \to \mathbb{D}_2$ is called monotonic, if $f(a) \sqsubseteq f(b)$ for all $a \sqsubseteq b$.