Script generated by TTT

Title: Seidl: Programmoptimierung (29.10.2012)

Date: Mon Oct 29 15:00:39 CET 2012

Duration: 89:32 min

Pages: 58

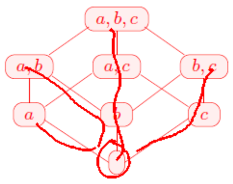
Background 2: Complete Lattices

A set $\mathbb{D}$ together with a relation $\sqsubseteq \subseteq \mathbb{D} \times \mathbb{D}$ is a partial order if for all $a, b, c \in \mathbb{D}$,

$$
\begin{array}{ll}
a \sqsubseteq a & \textit{reflexivity} \\
a \sqsubseteq b \wedge b \sqsubseteq a \implies a = b & \textit{anti-symmetry} \\
a \sqsubseteq b \wedge b \sqsubseteq c \implies a \sqsubseteq c & \textit{transitivity}
\end{array}
$$

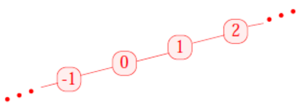
Examples:

1. $\mathbb{D} = 2^{\{a,b,c\}}$ with the relation "$\subseteq$" :

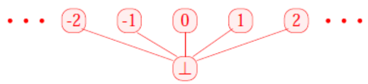


2. $\mathbb{Z}$ with the relation "$=$" :

3. $\mathbb{Z}$ with the relation "$\leq$" :



4. $\mathbb{Z}_\perp = \mathbb{Z} \cup \{\perp\}$ with the ordering:



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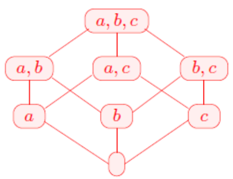

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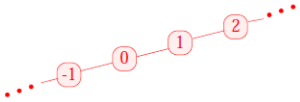

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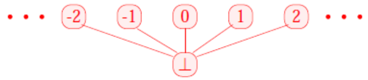

$a \sqsubseteq b$

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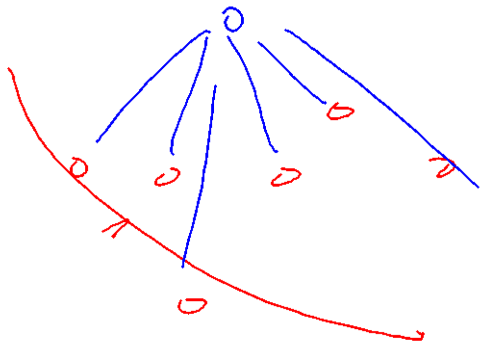
$$x \sqsubseteq d \qquad \text{for all } x \in X$$



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d is called least upper bound (lub) if

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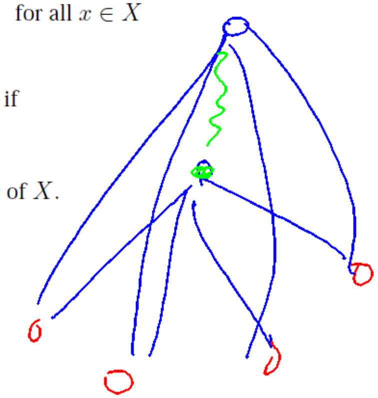

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- $\{0, 2, 4, \ldots\} \subseteq \mathbb{Z}$ has no upper bound!
- $\{0, 2, 4\} \subseteq \mathbb{Z}$ has the upper bounds $4, 5, 6, \ldots$

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A complete lattice (cl) $\mathbb{D}$ is a partial ordering where every subset $X \subseteq \mathbb{D}$ has a least upper bound $\bigsqcup X \in \mathbb{D}$.

Note:

Every complete lattice has

→ a least element $\bot = \bigsqcup \emptyset \in \mathbb{D}$;

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$\bigcup \{\{r\}, \{s\}, \emptyset\} = \{r, s\}$

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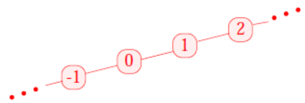

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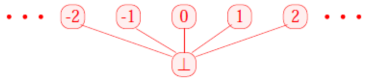
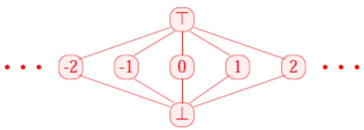


Examples:

1. $\mathbb{D} = 2^{\{a,b,c\}}$ is a cl :-)
2. $\mathbb{D} = \mathbb{Z}$ with "$=$" is not.
3. $\mathbb{D} = \mathbb{Z}$ with "$\leq$" is neither.
4. $\mathbb{D} = \mathbb{Z}_\perp$ is also not :-(
5. With an extra element $\top$, we obtain the flat lattice $\mathbb{Z}_\perp^\top = \mathbb{Z} \cup \{\perp, \top\}$:



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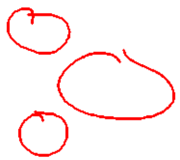
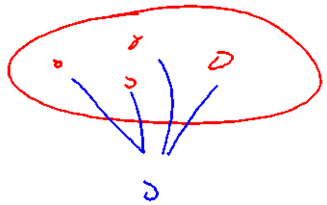


We have:

Theorem:

If $\mathbb{D}$ is a complete lattice, then every subset $X \subseteq \mathbb{D}$ has a greatest lower bound $\bigsqcap X$.



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Construct $U = \{u \in \mathbb{D} \mid \forall x \in X : \; u \sqsubseteq x\}$.

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Set: $g := \bigsqcup U$

Claim: $g = \bigsqcap X$

(1) g is a lower bound of X :

Assume $x \in X$. Then:

$u \sqsubseteq x$ for all $u \in U$

$\Longrightarrow$ x is an upper bound of U

$\Longrightarrow$ $g \sqsubseteq x$:-)

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(2) g is the greatest lower bound of X :

Assume u is a lower bound of X. Then:

$u \in U$

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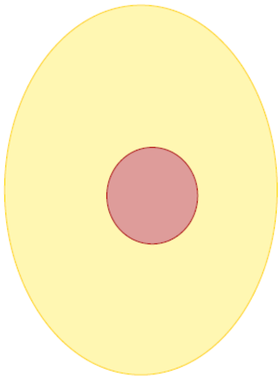

We are looking for solutions for systems of constraints of the form:

$$x_i \quad \sqsupseteq \quad f_i(x_1, \ldots, x_n) \qquad (*)$$

where:

x_i	unknown	here:	$\mathcal{A}[u]$
$\mathbb{D}$	values	here:	2^{Expr}
$\sqsubseteq \ \subseteq \ \mathbb{D} \times \mathbb{D}$	ordering relation	here:	$\supseteq$
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Constraint for $\mathcal{A}[v]$ $(v \neq start)$:

$$\mathcal{A}[v] \quad \subseteq \quad \bigcap\{[\![k]\!]^\sharp\,(\mathcal{A}[u]) \mid k = (u, \_, v) \;\; \text{edge}\}$$

$A[v] \subseteq [\![k_1]\!]^\#(A[u_1]) \cap$
$[\![k_2]\!]^\#(A[u_2])$

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Because:

$$x \sqsupseteq d_1 \wedge \ldots \wedge x \sqsupseteq d_k \quad \text{iff} \quad x \sqsupseteq \bigsqcup\{d_1, \dots, d_k\} \qquad :-)$$

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(1) $\mathbb{D}_1 = \mathbb{D}_2 = 2^U$ for a set U and $f\,x = (x \cap a) \cup b$.
Obviously, every such f is monotonic :-)

(2) $\mathbb{D}_1 = \mathbb{D}_2 = \mathbb{Z}$ (with the ordering "$\leq$"). Then:

- $\mathrm{inc}\, x = x + 1$ is monotonic.
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If $f_1 : \mathbb{D}_1 \to \mathbb{D}_2$ and $f_2 : \mathbb{D}_2 \to \mathbb{D}_3$ are monotonic, then also $f_2 \circ f_1 : \mathbb{D}_1 \to \mathbb{D}_3$:-)

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Theorem:

If $\mathbb{D}_2$ is a complete lattice, then the set $[\mathbb{D}_1 \to \mathbb{D}_2]$ of monotonic functions $f : \mathbb{D}_1 \to \mathbb{D}_2$ is also a complete lattice where

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In particular for $F \subseteq [\mathbb{D}_1 \to \mathbb{D}_2]$,

$$\bigsqcup F = f \quad \text{mit} \quad f\,x = \bigsqcup \{g\,x \mid g \in F\}$$

For functions $f_i\,x = a_i \cap x \cup b_i$, the operations "$\circ$", "$\sqcup$" and "$\sqcap$" can be explicitly defined by:

$$\begin{aligned}
(f_2 \circ f_1)\,x &= \boxed{a_1 \cap a_2} \cap x \cup \boxed{a_2 \cap b_1 \cup b_2} \\
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Wanted: minimally small solution for:

$$x_i \sqsupseteq f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n \qquad (*)$$

where all $f_i : \mathbb{D}^n \to \mathbb{D}$ are monotonic.

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Idea:

- Consider $F : \mathbb{D}^n \to \mathbb{D}^n$ where
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$$(1,2,3,1) \sqsubseteq (3,2,3,15)$$

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- We successively approximate a solution. We construct:

$$\bot, \quad F\,\bot, \quad F^2\,\bot, \quad F^3\,\bot, \quad \dots$$

Hope:

We eventually reach a solution ... ???

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Example: $\mathbb{D} = 2^{\{a,b,c\}}, \quad \sqsubseteq = \subseteq$

$$\begin{aligned} x_1 &\supseteq \{a\} \cup x_3 \\ x_2 &\supseteq x_3 \cap \{a, b\} \\ x_3 &\supseteq x_1 \cup \{c\} \end{aligned}$$

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The Iteration:

	0	1	2	3	4
x_1	$\emptyset$	a			
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x_1	$\emptyset$	$\{a\}$	$\{a,c\}$	$\{a,c\}$	dito
x_2	$\emptyset$	$\emptyset$	$\emptyset$	$\{a\}$	
x_3	$\emptyset$	$\{c\}$	$\{a,c\}$	$\{a,c\}$	

Theorem

- $\perp, F\perp, F^2\perp, \ldots$ form an ascending chain :

$$\perp \;\sqsubseteq\; F\perp \;\sqsubseteq\; F^2\perp \;\sqsubseteq\; \ldots$$

- If $F^k\perp = F^{k+1}\perp$, a solution is obtained which is the least one :-)
- If all ascending chains are finite, such a k always exists.

Example: $\mathbb{D} = 2^{\{a,b,c\}}, \quad \sqsubseteq = \subseteq$

$$\begin{array}{lll} x_1 & \supseteq & \{a\} \cup x_3 \\ x_2 & \supseteq & x_3 \cap \{a,b\} \\ x_3 & \supseteq & x_1 \cup \{c\} \end{array}$$

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Proof

The first claim follows by complete induction: $n = 0$

Foundation: $F^0\perp = \perp \sqsubseteq F^1\perp$:-)

Step: Assume $F^{i-1}\perp \sqsubseteq F^i\perp$. Then

$$F^i\perp = F\,(F^{i-1}\perp) \sqsubseteq \boxed{F\,(F^i\perp)} = \boxed{F^{i+1}\perp}$$

since F monotonic :-)

Theorem

- $\underline{\perp}, F\,\underline{\perp}, F^2\,\underline{\perp}, \ldots$ form an ascending chain :

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The first claim follows by complete induction:

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$\underline{d}$ any solution of

$X \underline{\sqsupseteq} F X \rightarrow$

$\underline{d} \underline{\sqsupseteq} F \underline{d}$

$\forall i.\ F^i\,\underline{\perp} \sqsubseteq \underline{d}$

$i=0 \quad \underline{\perp} \sqsubseteq \underline{d}$ ✓

$i>0 \quad F^{i-1}\,\underline{\perp} \sqsubseteq d \rightarrow F^i\,\underline{\perp} \sqsubseteq F\underline{d} \sqsubseteq \underline{d}$

Theorem

- $\underline{\perp}, F\,\underline{\perp}, F^2\,\underline{\perp}, \ldots$ form an ascending chain :

$$\underline{\perp} \;\sqsubseteq\; F\,\underline{\perp} \;\sqsubseteq\; F^2\,\underline{\perp} \;\sqsubseteq\; \ldots$$

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107

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since F monotonic :-)

Conclusion:

If $\mathbb{D}$ is finite, a solution can be found which is definitely the least :-)

Question:

What, if $\mathbb{D}$ is not finite ???

Theorem Knaster – Tarski

Assume $\mathbb{D}$ is a complete lattice. Then every monotonic function $f : \mathbb{D} \to \mathbb{D}$ has a least fixpoint $d_0 \in \mathbb{D}$.

Let $P = \{d \in \mathbb{D} \mid f\, d \sqsubseteq d\}$.

Then $d_0 = \bigsqcap P$.

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Assume $\mathbb{D}$ is a complete lattice. Then every monotonic function $f : \mathbb{D} \to \mathbb{D}$ has a least fixpoint $d_0 \in \mathbb{D}$.

Let $P = \{d \in \mathbb{D} \mid f\, d \sqsubseteq d\}$.

Then $d_0 = \bigsqcap P$.

$x \sqsupseteq f\, x$

Bronisław Knaster (1893-1980), topology