Script **generated by TTT**

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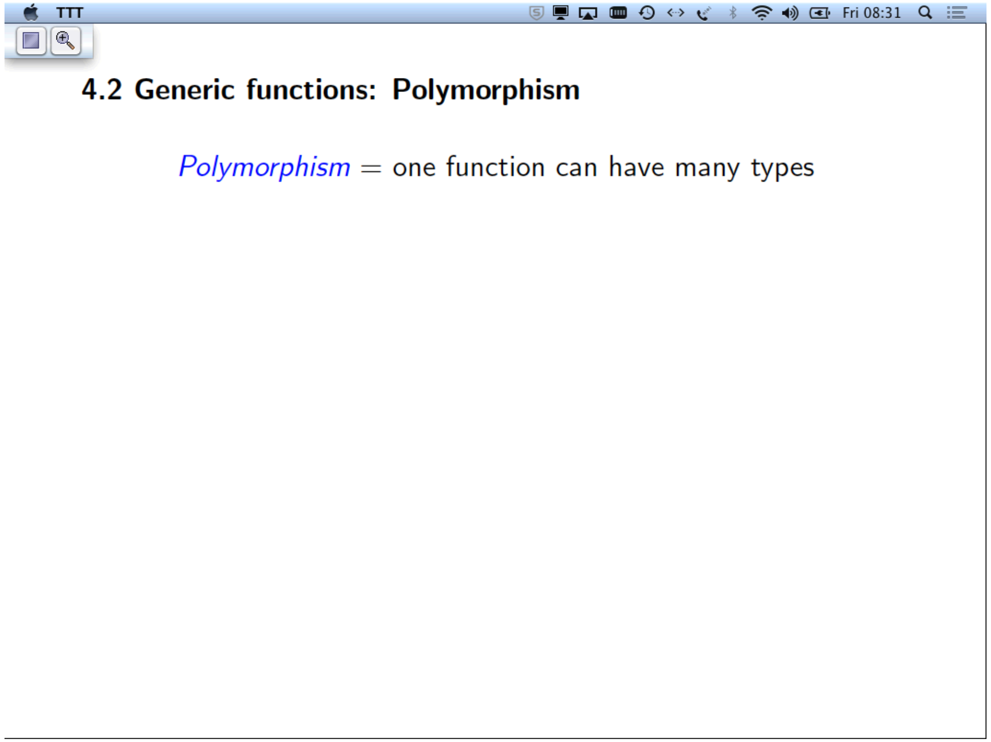
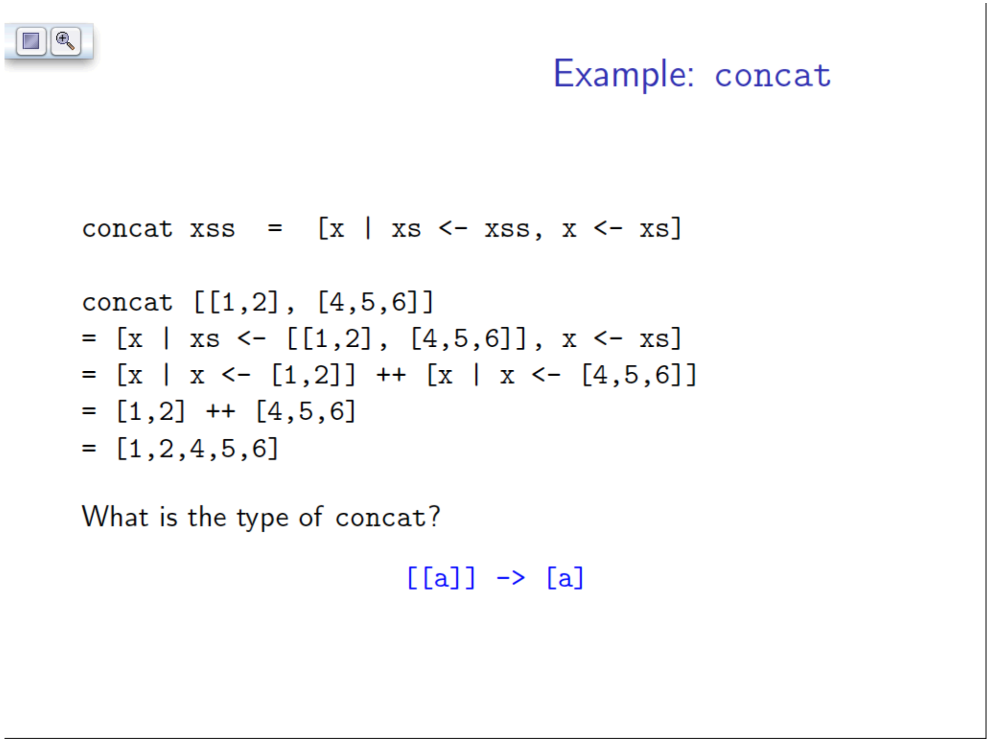


4.2 Generic functions: Polymorphism

Polymorphism = one function can have many types



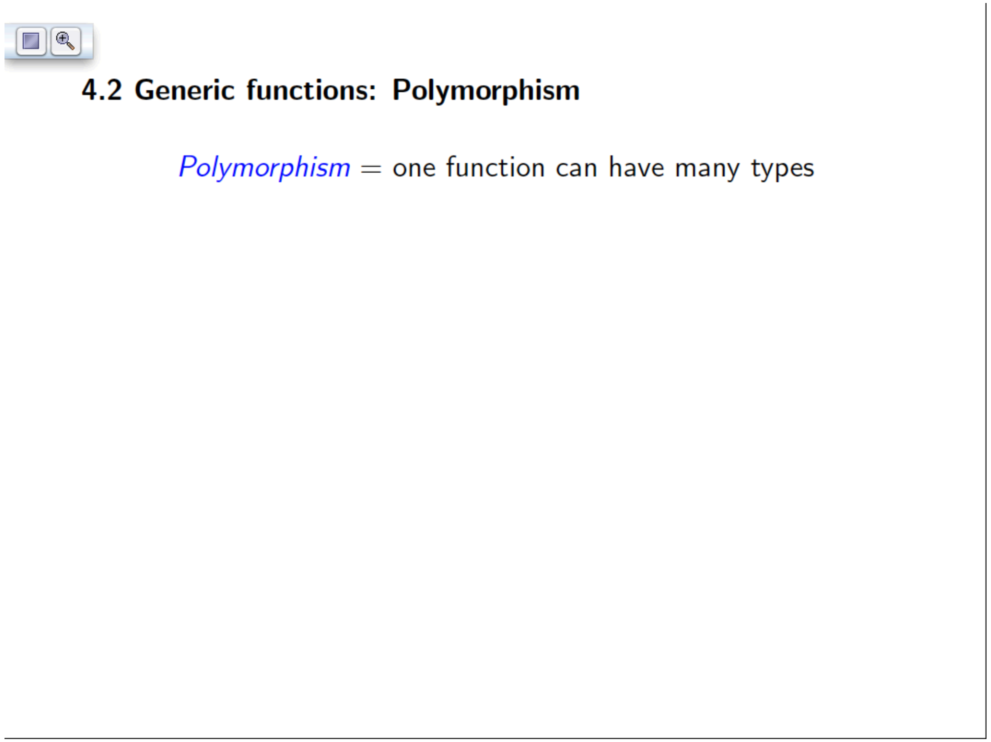
Example: `concat`

```
concat xss  =  [x | xs <- xss, x <- xs]

concat [[1,2], [4,5,6]]
= [x | xs <- [[1,2], [4,5,6]], x <- xs]
= [x | x <- [1,2]] ++ [x | x <- [4,5,6]]
= [1,2] ++ [4,5,6]
= [1,2,4,5,6]
```

What is the type of `concat`?

`[[a]] -> [a]`



4.2 Generic functions: Polymorphism

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Example

```
length :: [Bool] -> Int
length :: [Char] -> Int
```

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length :: [Char] -> Int
length :: [[Int]] -> Int
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⋮

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Example

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length :: [Bool] -> Int
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length :: [[Int]] -> Int
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⋮

The most general type:

`length :: [a] -> Int`

4.2 Generic functions: Polymorphism

Polymorphism = one function can have many types

Example

```
length :: [Bool] -> Int
length :: [Char] -> Int
length :: [[Int]] -> Int
```

⋮

The most general type:

`length :: [a] -> Int`

where `a` is a *type variable*

Type variable syntax

Type variables must start with a lower-case letter

Typically: `a`, `b`, `c`, . . .

Two kinds of polymorphism

Subtype polymorphism as in Java:

Two kinds of polymorphism

Subtype polymorphism as in Java:

$$\frac{f :: T \to U \qquad T' \leq T}{f :: T' \to U}$$

(remember: horizontal line = implication)

Parametric polymorphism as in Haskell:

Types may contain type variables ("parameters")

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where $T[U/a]$ = "T with a replaced by U"

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Example: $(a \to a)[Bool/a]$

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Example: $(a \to a)[Bool/a] \;=\; Bool \to Bool$

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$$\frac{f :: T}{f :: T[U/a]}$$

where $T[U/a]$ = "T with a replaced by U"

Example: $(a \to a)[Bool/a] \;=\; Bool \to Bool$

(Often called *ML-style polymorphism*)

Defining polymorphic functions

```
id :: a -> a
id x  =  x
```

Defining polymorphic functions

```
id :: a -> a
id x  =  x


fst (x,y)  =  x
```

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fst :: (a,b) -> a
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```

Defining polymorphic functions

```
id :: a -> a
id x  =  x

fst :: (a,b) -> a
fst (x,y)  =  x

swap :: (a,b) -> (b,a)
swap (x,y)  =  (y,x)
```

Defining polymorphic functions

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id :: a -> a
id x  =  x

fst :: (a,b) -> a
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silly :: Bool -> a -> Char
silly x y  =  if x then ’c’ else ’d’
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silly2 x y  =  if x then x else y
```

Defining polymorphic functions

```
id :: a -> a
id x  =  x

fst :: (a,b) -> a
fst (x,y)  =  x

swap :: (a,b) -> (b,a)
swap (x,y)  =  (y,x)

silly :: Bool -> a -> Char
silly x y  =  if x then ’c’ else ’d’

silly2 :: Bool -> Bool -> Bool
silly2 x y  =  if x then x else y
```

Polymorphic list functions from the Prelude

```
length :: [a] -> Int
length [5, 1, 9] = 3
```

Polymorphic list functions from the Prelude

```
length :: [a] -> Int
length [5, 1, 9] = 3

(++) :: [a] -> [a] -> [a]
[1, 2] ++ [3, 4] = [1, 2, 3, 4]

reverse :: [a] -> [a]
reverse [1, 2, 3] = [3, 2, 1]
```

Polymorphic list functions from the Prelude

```
length :: [a] -> Int
length [5, 1, 9] = 3

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[1, 2] ++ [3, 4] = [1, 2, 3, 4]

reverse :: [a] -> [a]
reverse [1, 2, 3] = [3, 2, 1]

replicate :: Int -> a -> [a]
replicate 3 ’c’ = "ccc"
```

Polymorphic list functions from the Prelude

```
head, last :: [a] -> a
head "list" = ’l’,
```

Polymorphic list functions from the Prelude

```
head, last :: [a] -> a
head "list" = ’l’,    last "list" = ’t’
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Polymorphic list functions from the Prelude

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head "list" = ’l’,    last "list" = ’t’

tail, init :: [a] -> [a]
tail "list" = "ist",
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Polymorphic list functions from the Prelude

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head, last :: [a] -> a
head "list" = ’l’,    last "list" = ’t’

tail, init :: [a] -> [a]
tail "list" = "ist",    init "list" = "lis"

take, drop :: Int -> [a] -> [a]
take 3 "list" = "lis",    drop 3 "list" = "t"
```

Polymorphic list functions from the Prelude

```
concat ::
concat [[1, 2], [3, 4], [0]] = [1, 2, 3, 4, 0]
```

Polymorphic list functions from the Prelude

```
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concat :: [[a]] -> [a]
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zip ::
zip [1,2] "ab" = [(1, ’a’), (2, ’b’)]
```

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concat :: [[a]] -> [a]
concat [[1, 2], [3, 4], [0]] = [1, 2, 3, 4, 0]

zip :: [a] -> [b] -> [(a,b)]
zip [1,2] "ab" = [(1, ’a’), (2, ’b’)]

unzip :: [(a,b)] -> ([a],[b])
unzip [(1, ’a’), (2, ’b’)] = ([1,2], "ab")
```

Polymorphic list functions from the Prelude

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concat :: [[a]] -> [a]
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-- A property
prop_zip xs ys =
  unzip(zip xs ys) ==
```

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Polymorphic list functions from the Prelude

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concat :: [[a]] -> [a]
concat [[1, 2], [3, 4], [0]] = [1, 2, 3, 4, 0]

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zip [1,2] "ab" = [(1, 'a'), (2, 'b')]

unzip :: [(a,b)] -> ([a],[b])
unzip [(1, 'a'), (2, 'b')] = ([1,2], "ab")

-- A property
prop_zip xs ys = length xs == length ys ==>
  unzip(zip xs ys) == (xs, ys)
```

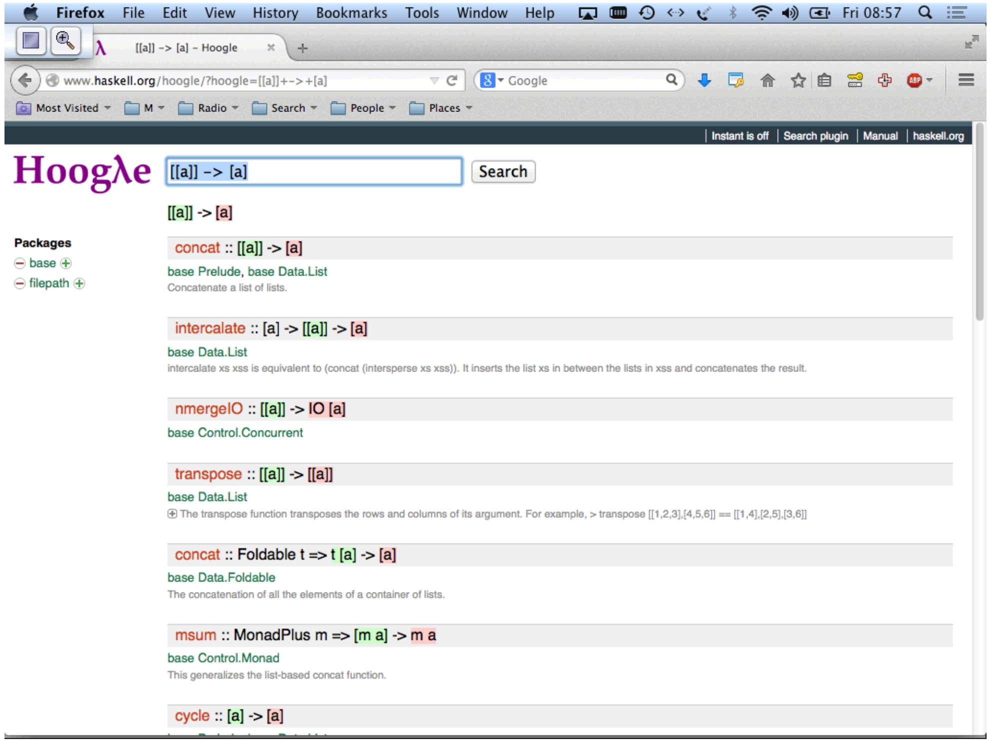
Haskell libraries

- Prelude and much more

Haskell libraries

- Prelude and much more
- Hoogle — searching the Haskell libraries

http://www.haskell.org/hoogle/



[[a]] -> [a]

Packages

⊖ base ⊕

⊖ filepath ⊕

concat :: [[a]] -> [a]

base Prelude, base Data.List

Concatenate a list of lists.

intercalate :: [a] -> [[a]] -> [a]

base Data.List

intercalate xs xss is equivalent to (concat (intersperse xs xss)). It inserts the list xs in between the lists in xss and concatenates the result.

nmergeIO :: [[a]] -> IO [a]

base Control.Concurrent

transpose :: [[a]] -> [[a]]

base Data.List

⊕ The transpose function transposes the rows and columns of its argument. For example, > transpose [[1,2,3],[4,5,6]] == [[1,4],[2,5],[3,6]]

concat :: Foldable t => t [a] -> [a]

base Data.Foldable

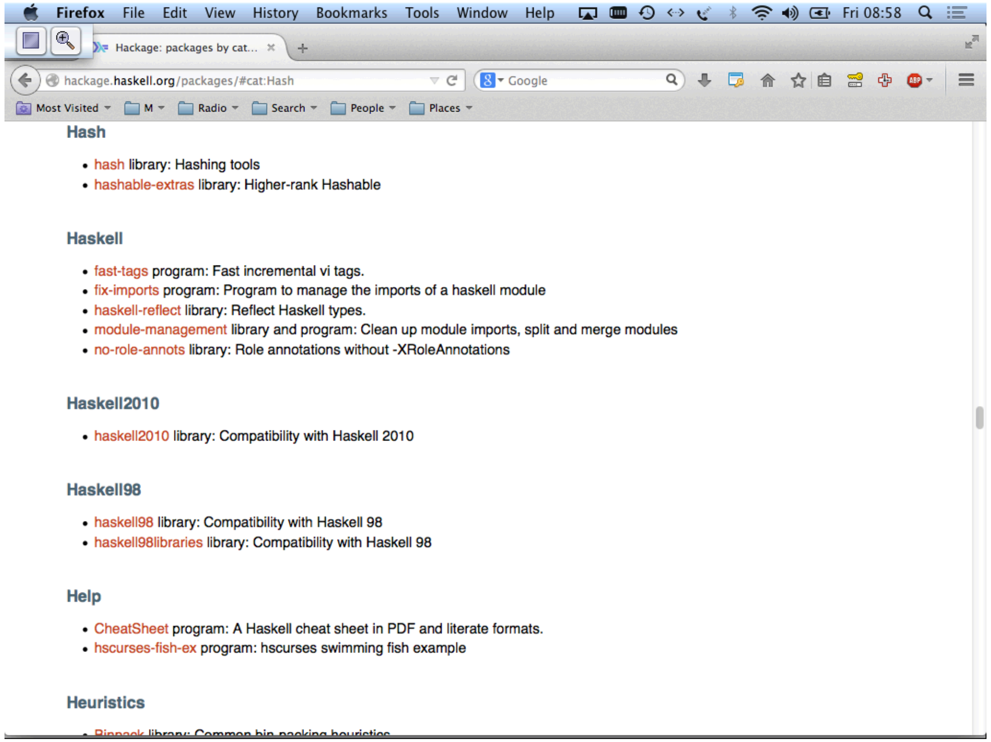
The concatenation of all the elements of a container of lists.

msum :: MonadPlus m => [m a] -> m a

base Control.Monad

This generalizes the list-based concat function.

cycle :: [a] -> [a]



Hash

- hash library: Hashing tools
- hashable-extras library: Higher-rank Hashable

Haskell

- fast-tags program: Fast incremental vi tags.
- fix-imports program: Program to manage the imports of a haskell module
- haskell-reflect library: Reflect Haskell types.
- module-management library and program: Clean up module imports, split and merge modules
- no-role-annots library: Role annotations without -XRoleAnnotations

Haskell2010

- haskell2010 library: Compatibility with Haskell 2010

Haskell98

- haskell98 library: Compatibility with Haskell 98
- haskell98libraries library: Compatibility with Haskell 98

Help

- CheatSheet program: A Haskell cheat sheet in PDF and literate formats.
- hscurses-fish-ex program: hscurses swimming fish example

Heuristics

- Binpack library: Common bin-packing heuristics

Haskell libraries

- Prelude and much more
- Hoogle — searching the Haskell libraries
- Hackage — a collection of Haskell packages

See Haskell pages and Thompson's book for more information.

Further list functions from the Prelude

```
and :: [Bool] -> Bool
and [True, False, True] = False

or :: [Bool] -> Bool
or [True, False, True] = True
```

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and :: [Bool] -> Bool
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-- For numeric types a:
sum, product :: [a] -> a
sum [1, 2, 2] = 5,
```

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and :: [Bool] -> Bool
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-- For numeric types a:
sum, product :: [a] -> a
sum [1, 2, 2] = 5,    product [1, 2, 2] = 4
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Further list functions from the Prelude

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or [True, False, True] = True

-- For numeric types a:
sum, product :: [a] -> a
sum [1, 2, 2] = 5,    product [1, 2, 2] = 4
```

What exactly is the type of `sum`, `prod`, `+`, `∗`, `==`, . . . ???

Polymorphism versus Overloading

Polymorphism: one definition, many types

Polymorphism versus Overloading

Polymorphism: one definition, many types

Overloading: different definition for different types

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Example

Function `(+)` is overloaded:

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Function `(+)` is overloaded:

- on type `Int`: built into the hardware

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Function `(+)` is overloaded:

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Example

Function `(+)` is overloaded:

- on type `Int`: built into the hardware
- on type `Integer`: realized in software

So what is the type of `(+)` ?

Numeric types

`(+) :: Num a => a -> a -> a`

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Function `(+)` has type `a -> a -> a` for any type of class `Num`

- Class `Num` is the class of *numeric types*.

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Function `(+)` has type `a -> a -> a` for any type of class `Num`

- Class `Num` is the class of *numeric types*.
- Predefined numeric types: `Int`, `Integer`, `Float`
- Types of class Num offer the basic arithmetic operations:

  ```
  (+) :: Num a => a -> a -> a
  (-) :: Num a => a -> a -> a
  (*) :: Num a => a -> a -> a
  ⋮
  ```

Other important type classes

Other important type classes

- The class `Eq` of *equality types*, i.e. types that posess

  ```
  (==) ::  Eq a => a -> a -> Bool
  (/=) ::  Eq a => a -> a -> Bool
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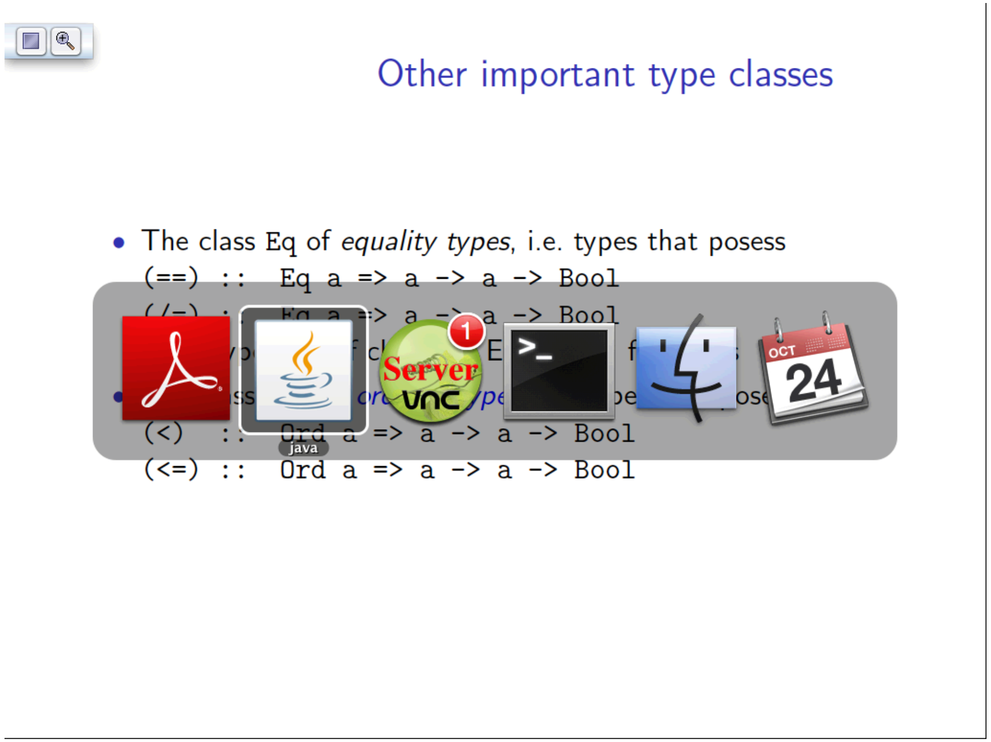
- The class `Ord` of *ordered types*, i.e. types that posess
  `(<)  ::  Ord a => a -> a -> Bool`



Other important type classes

- The class Eq of *equality types*, i.e. types that posess
  `(==) ::  Eq a => a -> a -> Bool`
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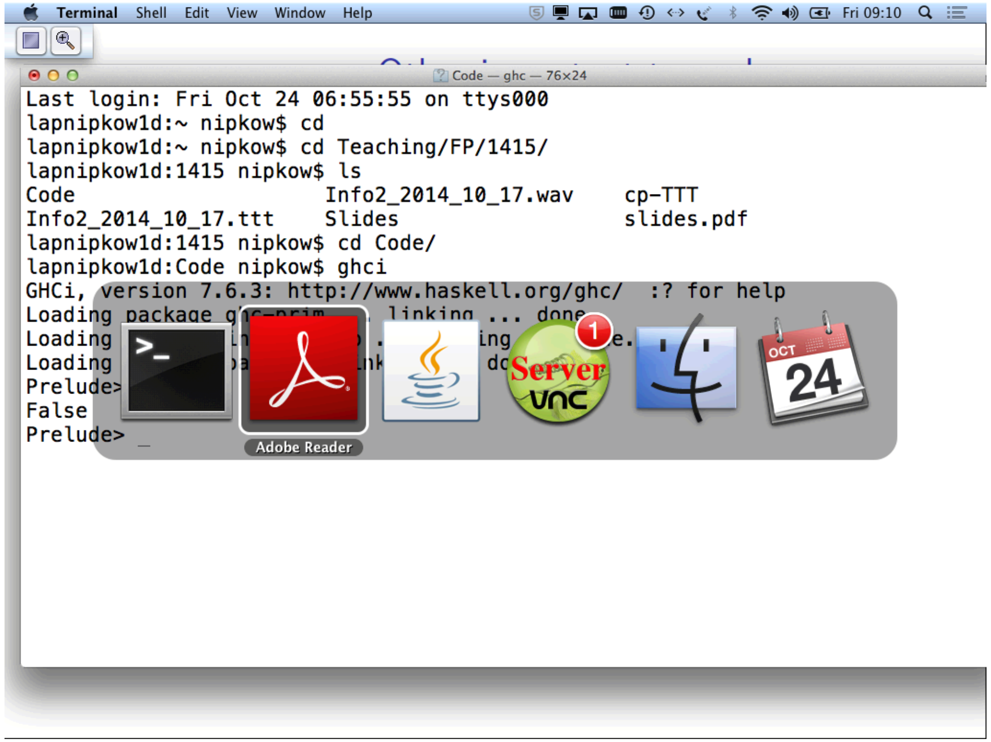

```
Last login: Fri Oct 24 06:55:55 on ttys000
lapnipkow1d:~ nipkow$ cd
lapnipkow1d:~ nipkow$ cd Teaching/FP/1415/
lapnipkow1d:1415 nipkow$ ls
Code                    Info2_2014_10_17.wav    cp-TTT
Info2_2014_10_17.ttt    Slides                  slides.pdf
lapnipkow1d:1415 nipkow$ cd Code/
lapnipkow1d:Code nipkow$ ghci
GHCi, version 7.6.3: http://www.haskell.org/ghc/  :? for help
Loading package ghc-prim ... linking ... done.
Loading
Loading
Prelude>
False
Prelude>
```

Other important type classes

- The class Eq of *equality types*, i.e. types that posess
  `(==) ::  Eq a => a -> a -> Bool`
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- The class `Ord` of *ordered types*, i.e. types that posess
  ```
  (<)  ::  Ord a => a -> a -> Bool
  (<=) ::  Ord a => a -> a -> Bool
  ```

More on type classes later. Don't confuse with OO classes.

Warning: == []

Warning: == []

```
null xs  =  xs == []
```

Warning: == []

```
null :: [a] -> Bool
null xs  =  xs == []
```

Warning: == []

```
null :: Eq a => [a] -> Bool
null xs  =  xs == []
```

Warning: == []

```
null :: Eq a => [a] -> Bool
null xs  =  xs == []
```

Why?

== on [a] may call == on a

Better:

```
null :: [a] -> Bool
null []  =  True
null _   =  False
```

In Prelude!

Warning: QuickCheck and polymorphism

QuickCheck does not work well on polymorphic properties

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QuickCheck does not work well on polymorphic properties

Example

QuickCheck does not find a counterexample to

```
prop_reverse :: [a] -> Bool
prop_reverse xs  =  reverse xs == xs
```

The solution: specialize the polymorphic property, e.g.

```
prop_reverse :: [Int] -> Bool
prop_reverse xs  =  reverse xs == xs
```

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prop_reverse :: [Int] -> Bool
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```

Now QuickCheck works

Conditional properties have result type `Property`

4.3 Case study: Pictures

```
type Picture = [String]
```

Conditional properties have result type `Property`

Example

```
prop_rev10 :: [Int] -> Property
prop_rev10 xs =
  length xs <= 10 ==> reverse(reverse xs) == xs
```

Conditional properties have result type `Property`

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Example

```
prop_rev10 :: [Int] -> Property
prop_rev10 xs =
  length xs <= 10 ==> reverse(reverse xs) == xs
```

4.3 Case study: Pictures

```
type Picture = [String]
```

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```
type Picture = [String]

uarr ::  Picture
uarr =
 ["  #  ",
  " ### ",
  "#####",
  "  #  ",
  "  #  ",
```

```
flipH :: Picture -> Picture
```

4.3 Case study: Pictures

```
type Picture = [String]

uarr ::  Picture             larr ::  Picture
uarr =                       larr =
 ["  #  ",                    ["  #  ",
  " ### ",                     " ##  ",
  "#####",                     "#####",
  "  #  ",                     " ##  ",
  "  #  ",                     "  #  ",
```

```
flipH :: Picture -> Picture
flipH  =  reverse
```

```
flipH :: Picture -> Picture
flipH  =  reverse

flipV :: Picture -> Picture
flipV pic  =
```

```
flipH :: Picture -> Picture
flipH  =  reverse

flipV :: Picture -> Picture
flipV pic  =  [ reverse line | line <- pic]

rarr :: Picture
rarr  =  flipV larr

darr :: Picture
darr  =  flipH uarr
```

```
flipH :: Picture -> Picture
flipH  =  reverse

flipV :: Picture -> Picture
flipV pic  =  [ reverse line | line <- pic]

rarr :: Picture
rarr  =  flipV larr

darr :: Picture
darr  =  flipH uarr

above :: Picture -> Picture -> Picture
above  =
```

```
flipH :: Picture -> Picture
flipH  =  reverse

flipV :: Picture -> Picture
flipV pic  =  [ reverse line | line <- pic]

rarr :: Picture
rarr  =  flipV larr

darr :: Picture
darr  =  flipH uarr

above :: Picture -> Picture -> Picture
above  =  (++)
```

```
flipH :: Picture -> Picture
flipH  =  reverse

flipV :: Picture -> Picture
flipV pic  =  [ reverse line | line <- pic]

rarr :: Picture
rarr  =  flipV larr

darr :: Picture
darr  =  flipH uarr

above :: Picture -> Picture -> Picture
above  =  (++)

beside :: Picture -> Picture -> Picture
```

```
flipH :: Picture -> Picture
flipH  =  reverse

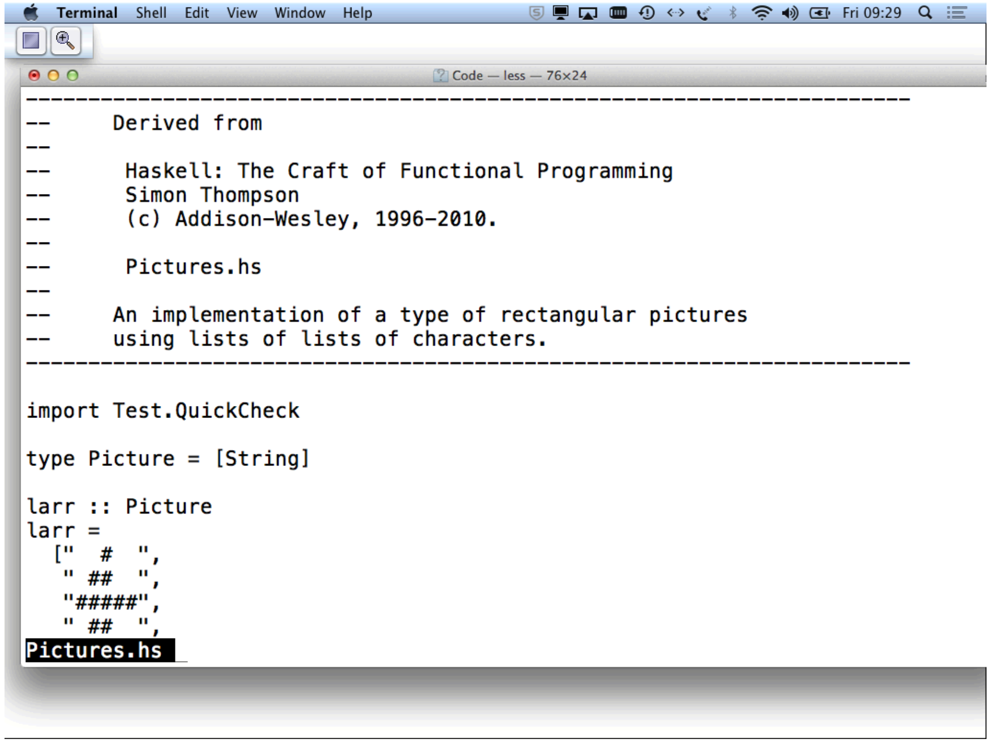
flipV :: Picture -> Picture
flipV pic  =  [ reverse line | line <- pic]

rarr :: Picture
rarr  =  flipV larr

darr :: Picture
darr  =  flipH uarr

above :: Picture -> Picture -> Picture
above  =  (++)

beside :: Picture -> Picture -> Picture
beside pic1 pic2 =
```

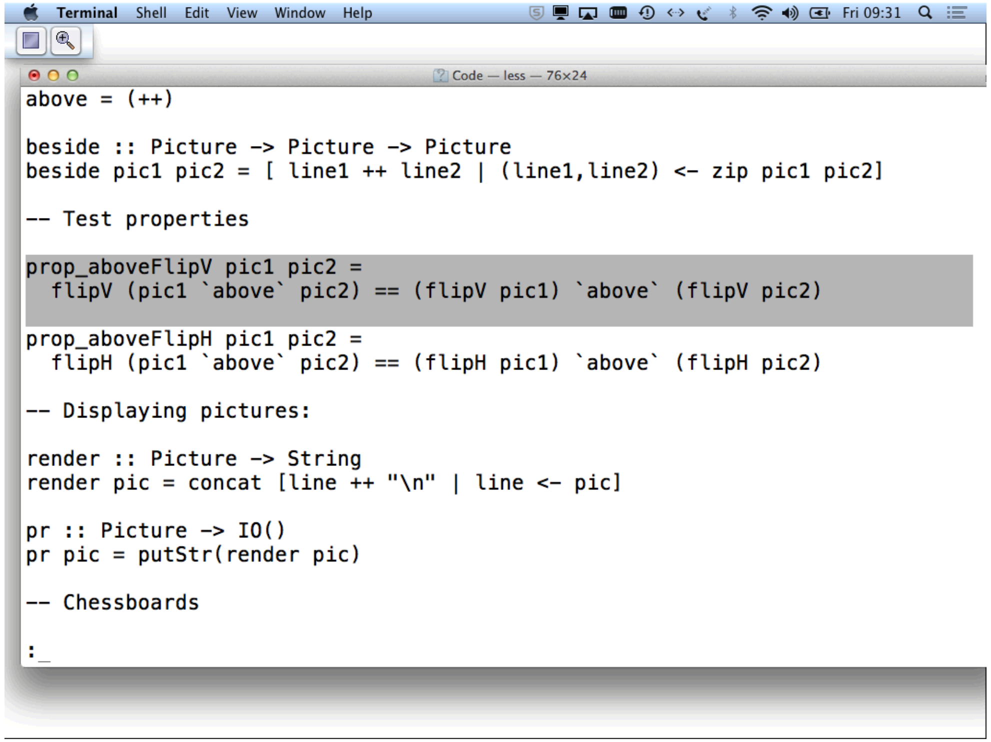


```
------------------------------------------------------------------------
--     Derived from
--
--      Haskell: The Craft of Functional Programming
--      Simon Thompson
--      (c) Addison-Wesley, 1996-2010.
--
--      Pictures.hs
--
--     An implementation of a type of rectangular pictures
--     using lists of lists of characters.
------------------------------------------------------------------------

import Test.QuickCheck

type Picture = [String]

larr :: Picture
larr =
  ["  #  ",
   " ##  ",
   "#####",
   " ##  ",
Pictures.hs
```



```
above = (++)

beside :: Picture -> Picture -> Picture
beside pic1 pic2 = [ line1 ++ line2 | (line1,line2) <- zip pic1 pic2]

-- Test properties

prop_aboveFlipV pic1 pic2 =
  flipV (pic1 `above` pic2) == (flipV pic1) `above` (flipV pic2)

prop_aboveFlipH pic1 pic2 =
  flipH (pic1 `above` pic2) == (flipH pic1) `above` (flipH pic2)

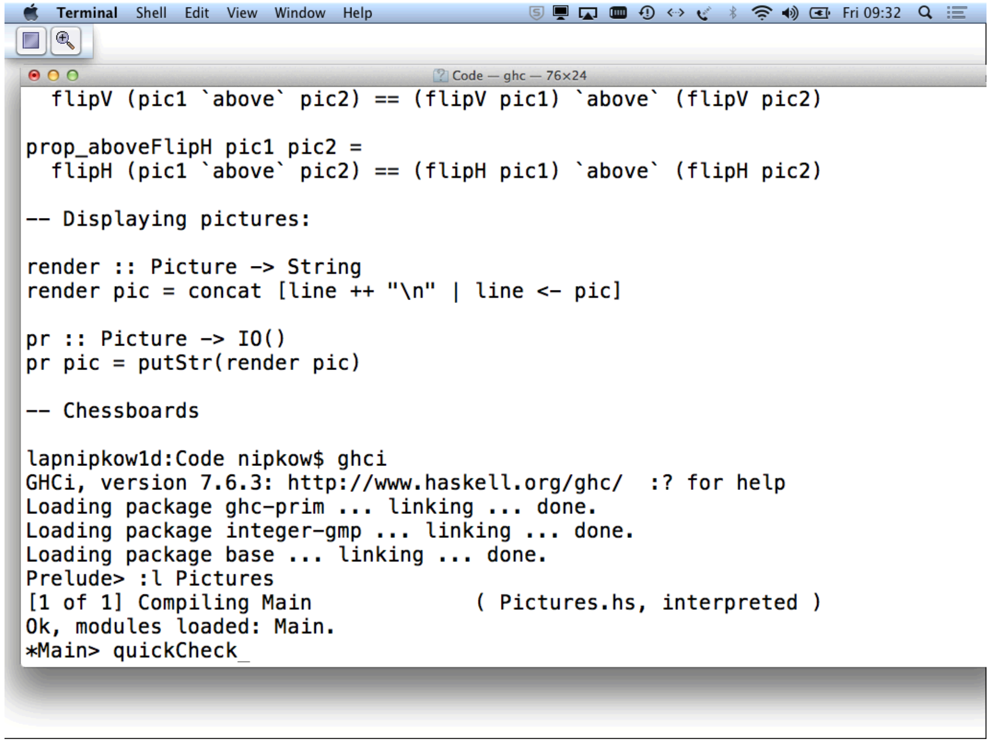
-- Displaying pictures:

render :: Picture -> String
render pic = concat [line ++ "\n" | line <- pic]

pr :: Picture -> IO()
pr pic = putStr(render pic)

-- Chessboards

:_
```



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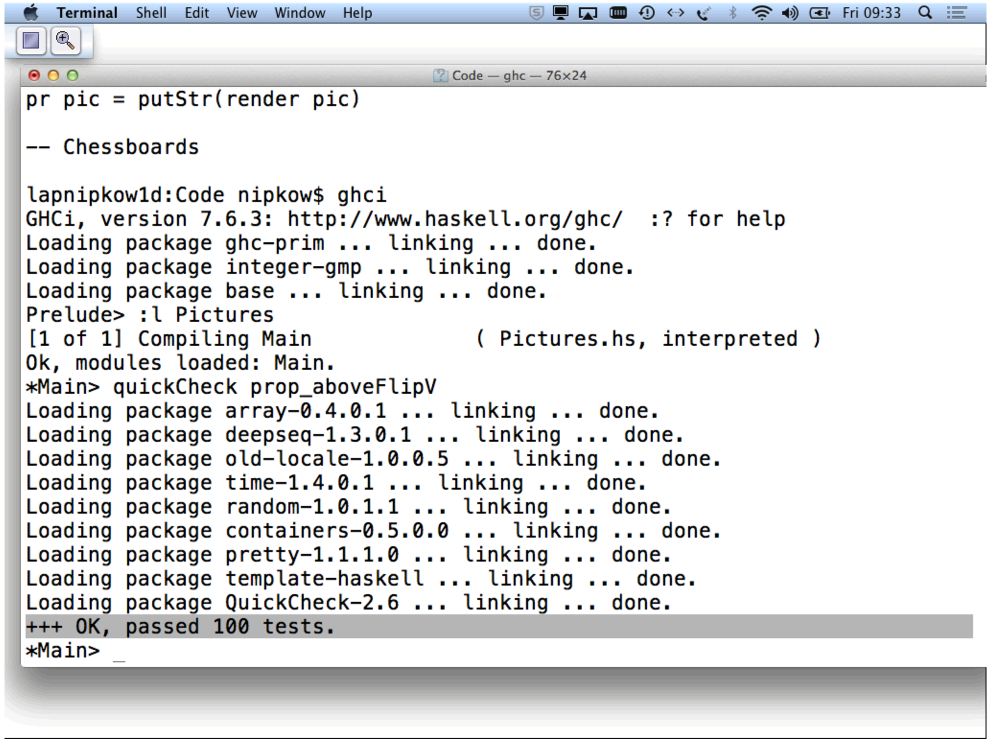
Prelude> :l Pictures
[1 of 1] Compiling Main             ( Pictures.hs, interpreted )
Ok, modules loaded: Main.
*Main> quickCheck_
```



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+++ OK, passed 100 tests.
*Main> _
```

Chessboards

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```
bSq  =  replicate 5 (replicate 5 ’#’)
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```

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alterH pic1 pic2 n  =
```

Chessboards

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alterH pic1 pic2 n  =  pic1 ‘beside‘ alterH pic2 pic1 (n-1
```

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alterV :: Picture -> Picture -> Int -> Picture
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alterV pic1 pic2 n  =  pic1 ‘above‘ alterV pic2 pic1 (n-1)

chessboard :: Int -> Picture
```

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```

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chessboard :: Int -> Picture
chessboard n  =  alterV bw wb n where
  bw = alterH bSq wSq n
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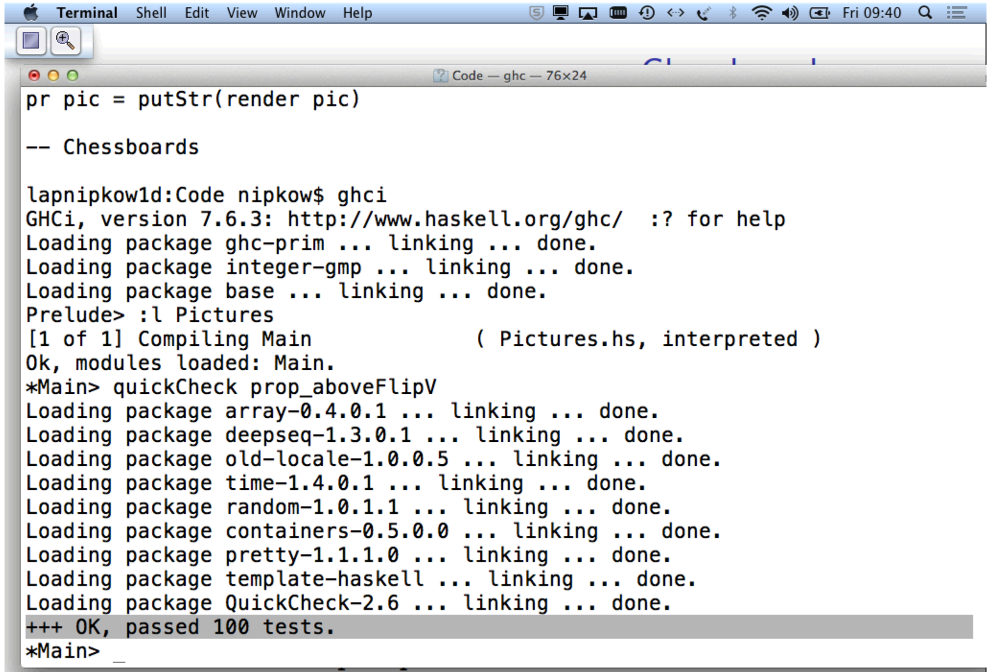

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```
  wb = alterH wSq bSq n
```

Exercise

Ensure that the lower left square of `chesboard n` is always black.

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$[x_1, \ldots, x_n]$	$x_1 : \ldots : x_n :$ `[]`

Note: $x : y : zs = x : (y : zs)$
`(:)` associates to the right

$\Longrightarrow$

Every list is either

`[]` or of the form

$x : xs$

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Every non-empty list can be decomposed uniquely into head and tail.

Therefore these definitions make sense:

```
head (x : xs)  =  x
tail (x : xs)  =  xs
```

`(++)` is not a constructor:

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- a constructor pattern $C\ p_1\ \ldots\ p_n$
  where C is a constructor and each p_i is a pattern

Note: `True` and `False` are constructors, too!

Function definitions by pattern matching

Example

```
head :: [a] -> a
head (x : _)  =  x
```

Function definitions by pattern matching

Example

```
head :: [a] -> a
head (x : _)  =  x

tail :: [a] -> [a]
tail (_ : xs)  =  xs
```

Function definitions by pattern matching

Example

```
head :: [a] -> a
head (x : _)  =  x

tail :: [a] -> [a]
tail (_ : xs)  =  xs

null :: [a] -> Bool
null []        =  True
null (_ : _)   =  False
```

Function definitions by pattern matching

$$\begin{array}{lcl} f\ pat_1 & = & e_1 \\ \vdots & & \\ f\ pat_n & = & e_n \end{array}$$

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Conditional equations:

$$f\ \ patterns\ \mid\ condition\ \ =\ \ e$$

When f is called, the equations are tried in the given order

Function definitions by pattern matching

Example (contrived)

```
true12 (True : True : _)  =  True
true12 _  =  False
```

Function definitions by pattern matching

Example (contrived)

```
true12 :: [Bool] -> Bool
true12 (True : True : _)  =  True
true12 _  =  False

same12 (x : _) (_ : y : _)  =  x == y
```

Function definitions by pattern matching

Example (contrived)

```
true12 :: [Bool] -> Bool
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asc3 (x : y : z : _)  =  x < y && y < z
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same12 (x : _) (_ : y : _)  =  x == y


asc3 (x : y : z : _)  =  x < y && y < z
asc3 (x : y : _)  =  x < y
asc3 _  =  True
```

Function definitions by pattern matching

Example (contrived)

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true12 :: [Bool] -> Bool
true12 (True : True : _)  =  True
true12 _  =  False

same12 :: Eq a => [a] -> [a] -> Bool
same12 (x : _) (_ : y : _)  =  x == y

asc3 :: Ord a => [a] -> Bool
asc3 (x : y : z : _)  =  x < y && y < z
asc3 (x : y : _)  =  x < y
asc3 _  =  True
```

4.5 Recursion over lists

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Example

```
length []          =  0
```

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```
length []          =  0
length (_ : xs)    =  length xs + 1

reverse []         =  []
```

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length []          =  0
length (_ : xs)    =  length xs + 1

reverse []         =  []
reverse (x : xs)  =
```

Primitive recursion on lists:

```
f []          =  base     -- base case
f (x : xs)    =  rec      -- recursive case
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- *base*: no call of *f*

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Primitive recursion on lists:

```
f []         =  base     -- base case
f (x : xs)   =  rec      -- recursive case
```

- *base*: no call of *f*
- *rec*: only call(s) *f xs*

f may have additional parameters.

Finding primitive recursive definitions

Example

```
concat :: [[a]] -> [a]
concat []  =  []
```

Finding primitive recursive definitions

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concat :: [[a]] -> [a]
concat []  =  []
concat (xs : xss)  =  xs ++ concat xss
```

Finding primitive recursive definitions

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concat :: [[a]] -> [a]
concat []  =  []
concat (xs : xss)  =  xs ++ concat xss

(++) :: [a] -> [a] -> [a]
[] ++ ys  =  ys
```

Insertion sort

Example

```
inSort :: [a] -> [a]
inSort []      =  []
inSort (x:xs)  =
```

Insertion sort

Example

```
inSort :: [a] -> [a]
inSort []      =  []
inSort (x:xs)  =        (inSort xs)
```

Insertion sort

Example

```
inSort :: [a] -> [a]
inSort []      =  []
inSort (x:xs)  =  ins x (inSort xs)

ins :: a -> [a] -> [a]
ins x []  =  [x]
ins x (y:ys)
```

Insertion sort

Example

```
inSort :: [a] -> [a]
inSort []       =  []
inSort (x:xs)  =  ins x (inSort xs)

ins :: a -> [a] -> [a]
ins x []  =  [x]
ins x (y:ys) | x <= y      =
```

Insertion sort

Example

```
inSort :: [a] -> [a]
inSort []       =  []
inSort (x:xs)  =  ins x (inSort xs)

ins :: a -> [a] -> [a]
ins x []  =  [x]
ins x (y:ys) | x <= y      =  x : y : ys
             | otherwise  =
```

Insertion sort

Example

```
inSort :: [a] -> [a]
inSort []       =  []
inSort (x:xs)  =  ins x (inSort xs)

ins :: a -> [a] -> [a]
ins x []  =  [x]
ins x (y:ys) | x <= y      =  x : y : ys
             | otherwise  =  y : ins x ys
```

Beyond primitive recursion: Complex patterns

Example

```
ascending :: Ord a => [a] -> bool
ascending []  =  True
ascending [_]  =  True
ascending (x : y : zs)  =
```

Beyond primitive recursion: Complex patterns

Example

```
ascending :: Ord a => [a] -> bool
ascending []  =  True
ascending [_]  =  True
ascending (x : y : zs)  =  x <= y && ascending (y : ys)
```

Beyond primitive recursion: Multiple arguments

Example

```
zip :: [a] -> [b] -> [(a,b)]
zip (x:xs) (y:ys)  =  (x,y) : zip xs ys
```

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zip _ _   =
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zip _ _   =  []
```

Beyond primitive recursion: Multiple arguments

Example

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zip :: [a] -> [b] -> [(a,b)]
zip (x:xs) (y:ys)  =  (x,y) : zip xs ys
zip _ _    =  []
```

Alternative definition:

```
zip’ [] []  =  []
zip’ (x:xs) (y:ys)  =  (x,y) : zip’ xs ys
```

`zip’` is undefined for lists of different length!

Beyond primitive recursion: Multiple arguments

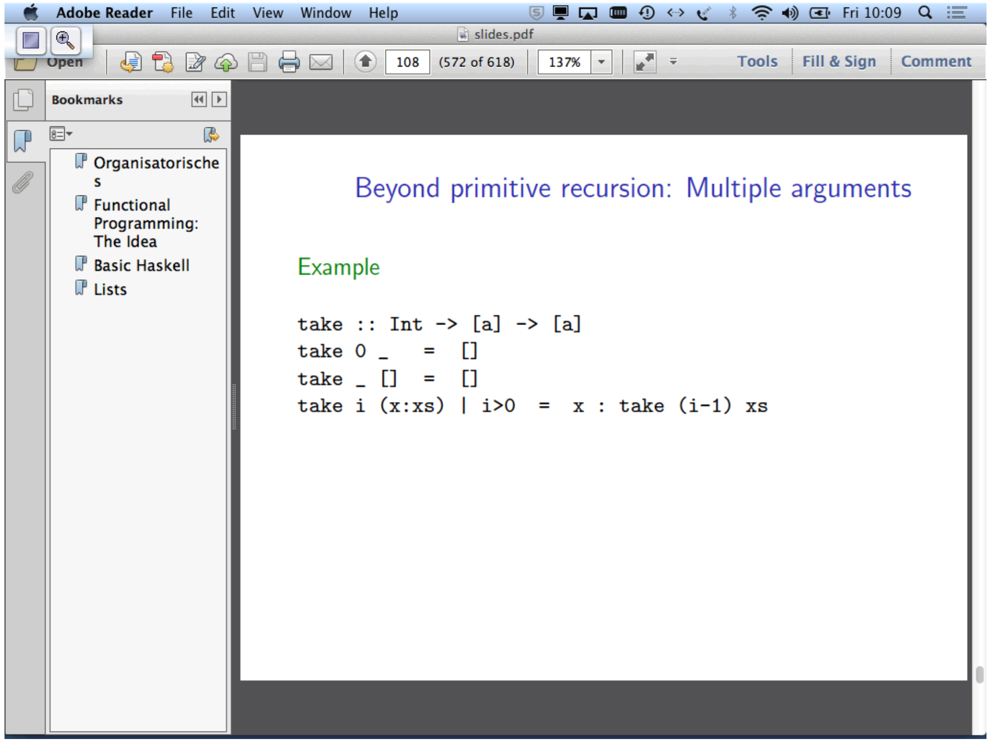
Example

```
take :: Int -> [a] -> [a]
take 0 _   =  []
take _ []  =  []
```

General recursion: Quicksort

Example

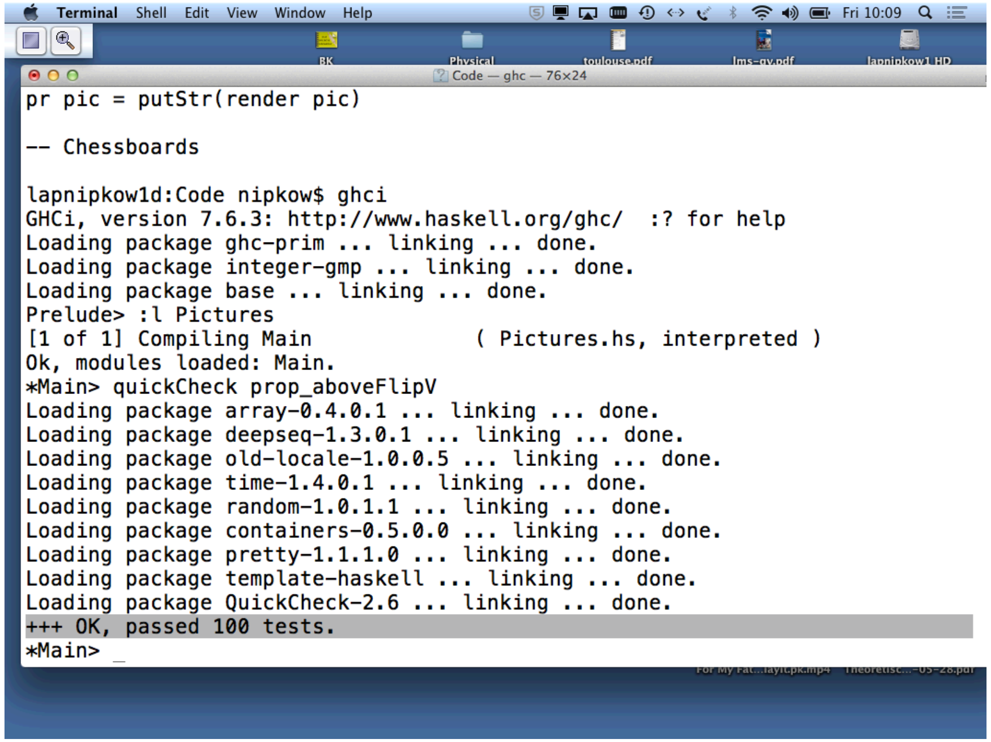
```
quicksort :: Ord a => [a] -> [a]
```



Beyond primitive recursion: Multiple arguments

Example

```
take :: Int -> [a] -> [a]
take 0 _   =  []
take _ []  =  []
take i (x:xs) | i>0  =  x : take (i-1) xs
```



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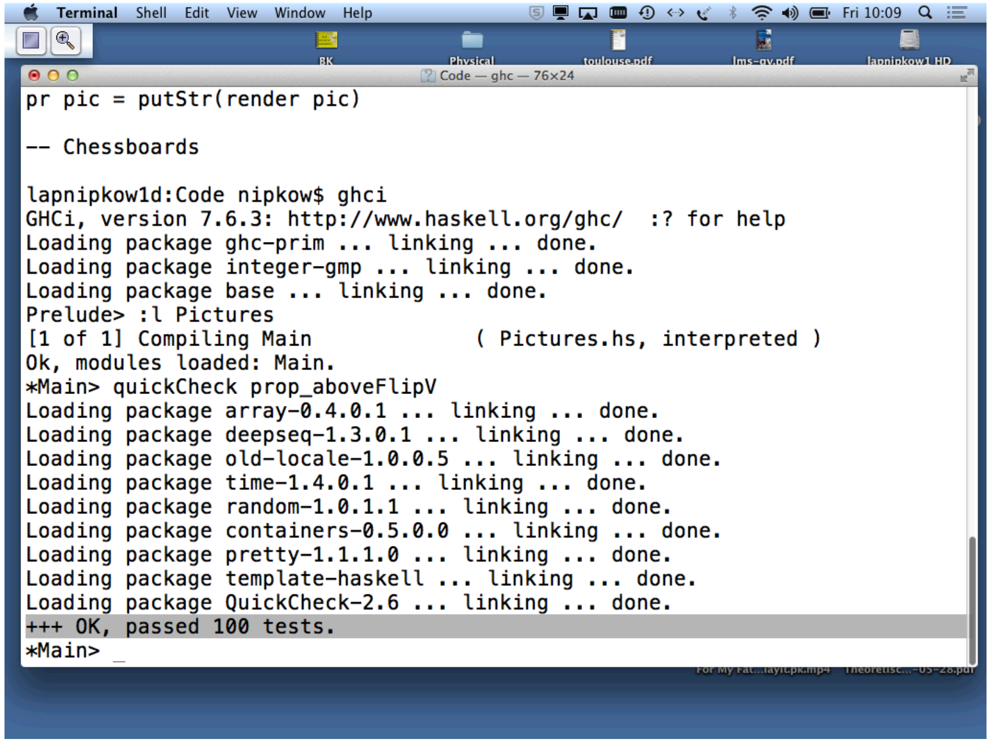
Loading package old-locale-1.0.0.5 ... linking ... done.
Loading package time-1.4.0.1 ... linking ... done.
Loading package random-1.0.1.1 ... linking ... done.
Loading package containers-0.5.0.0 ... linking ... done.
Loading package pretty-1.1.1.0 ... linking ... done.
Loading package template-haskell ... linking ... done.
Loading package QuickCheck-2.6 ... linking ... done.
+++ OK, passed 100 tests.
*Main> _
```



```
pr pic = putStr(render pic)

-- Chessboards

lapnipkow1d:Code nipkow$ ghci
GHCi, version 7.6.3: http://www.haskell.org/ghc/  :? for help
Loading package ghc-prim ... linking ... done.
Loading package integer-gmp ... linking ... done.
Loading package base ... linking ... done.
Prelude> :l Pictures
[1 of 1] Compiling Main             ( Pictures.hs, interpreted )
Ok, modules loaded: Main.
*Main> quickCheck prop_aboveFlipV
Loading package array-0.4.0.1 ... linking ... done.
Loading package deepseq-1.3.0.1 ... linking ... done.
Loading package old-locale-1.0.0.5 ... linking ... done.
Loading package time-1.4.0.1 ... linking ... done.
Loading package random-1.0.1.1 ... linking ... done.
Loading package containers-0.5.0.0 ... linking ... done.
Loading package pretty-1.1.1.0 ... linking ... done.
Loading package template-haskell ... linking ... done.
Loading package QuickCheck-2.6 ... linking ... done.
+++ OK, passed 100 tests.
*Main> _
```