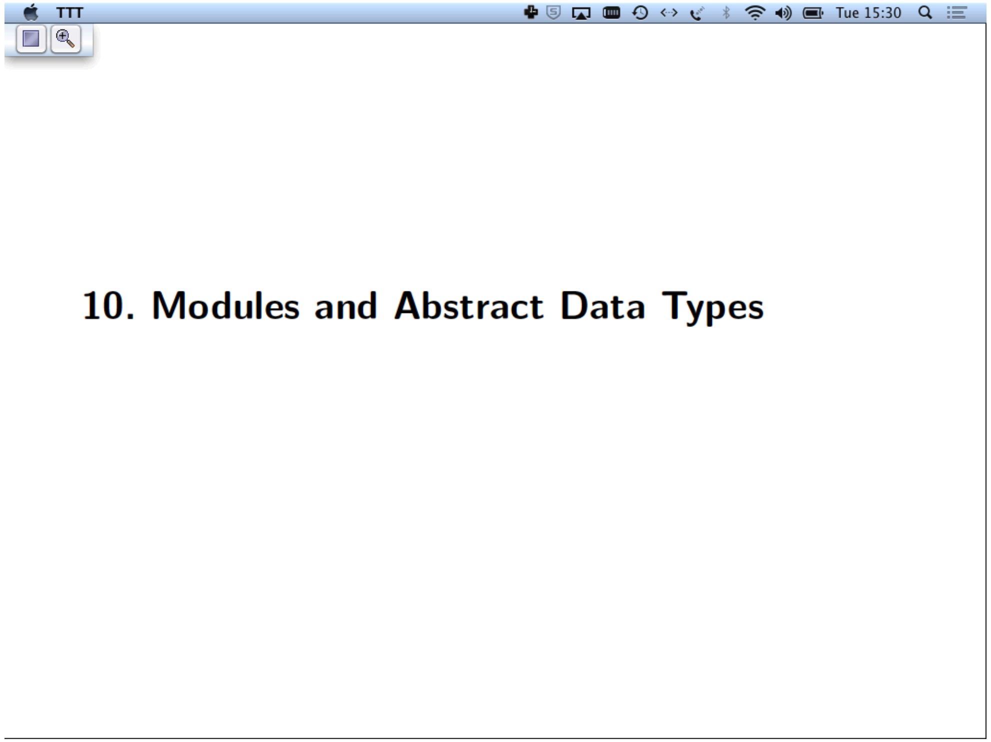
Script generated by TTT

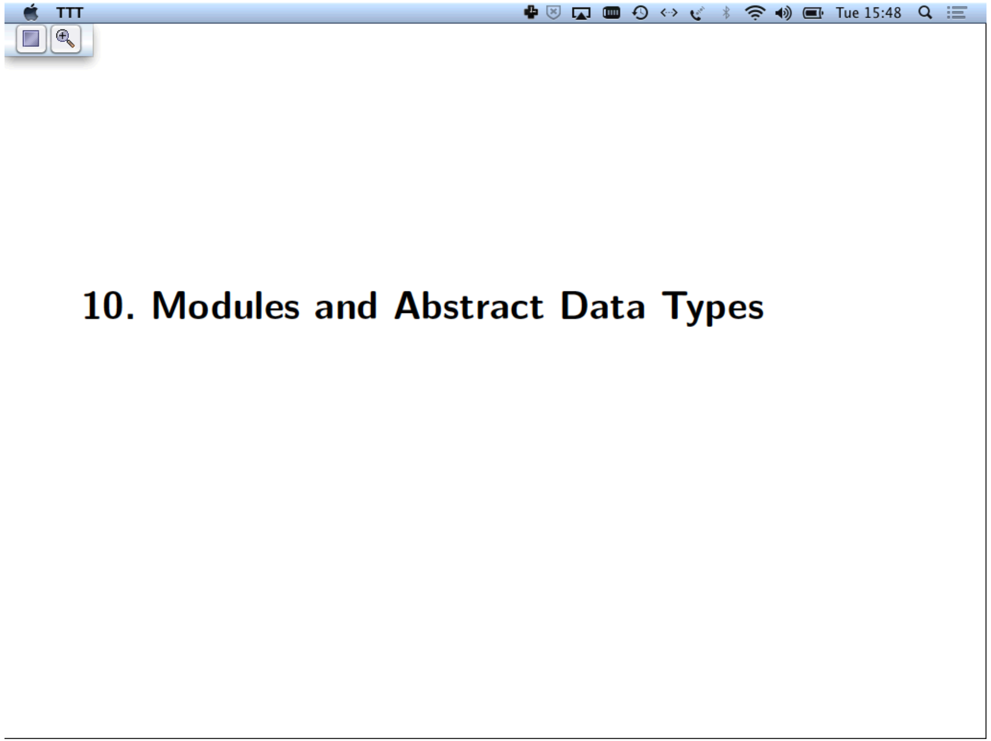
Title: Nipkow: Info2 (07.01.2014)

Date: Tue Jan 07 15:30:35 CET 2014

Duration: 88:46 min

Pages: 119





10.1 Modules

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Purposes:

- Grouping
- Interfaces
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GHC: one module per file

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- Interfaces
- Division of labour
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GHC: one module per file

Recommendation: module `M` in file `M.hs`

Module header

```
module M where    -- M must start with capital letter
↑
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All definitions must start in this column

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Selective export:

```
module M (T, f, ...)  where
```

Exporting data types

```
module M (T) where
data T = ...
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- Exports T and its constructors C, D, ...

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module M (T(..)) where
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- Exports T and all of its constructors

Not permitted: `module M (T,C,D) where`

Exporting modules

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```
module B where
import A
...
```

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⟹ B does not export f

Unless the names are mentioned in the export list

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Or the whole module is exported

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module B (module A) where
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```

import

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module B where          module A where
import A                f = ...
...                     g = ...
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⟹ B imports only f

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```

⟹ B imports f and g

Unless an import list is specified

```
module B where
import A (f)
...
```

⟹ B imports only f

Or specific names are hidden

```
module B where
import A hiding (g)
...
```

qualified

```
import A
import B
import C
... f ...
```

Where does f come from??

qualified

```
import A
import B
import C
... f ...
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Where does `f` come from??

Clearer: *qualified names*

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... A.f ...
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Where does `f` come from??

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Can be enforced:

```
import qualified A
```

Renaming modules

```
import TotallyAwesomeModule

... TotallyAwesomeModule.f ...
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Painful

More readable:

```
import qualified TotallyAwesomeModule as TAM
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For the full description of the module system
see the Haskell report

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... TotallyAwesomeModule.f ...
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... TAM.f ...
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Abstract Data Types do not expose their internal representation

Why? Example: sets implemented as lists without duplicates

- Could create illegal value: `[1, 1]`
- Could distinguish what should be indistinguishable: `[1, 2] /= [2, 1]`
- Cannot easily change representation later

Example: Sets

```
module Set where
-- sets are represented as lists w/o duplicates
type Set a = [a]
empty  :: Set a
empty  =  []
insert :: a -> Set a -> Set a
insert x xs  =  ...
isin :: a -> Set a -> Set a
isin x xs  =  ...
size :: Set a -> Integer
size xs  =  ...
```


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Exposes everything

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Exposes everything

Allows nonsense like `Set.size [1,1]`

Better

```
module Set (Set, empty, insert, isin, size) where
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```
module Set (Set, empty, insert, isin, size) where
-- Interface
empty  :: Set a
insert :: Eq a => a -> Set a -> Set a
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Better

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module Set (Set, empty, insert, isin, size) where
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- Explicit export list/interface
- But representation still not hidden

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type Set a = [a]
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- Explicit export list/interface
- But representation still not hidden
  Does not help: hiding the type name Set

Hiding the representation

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module Set (Set, empty, insert, isin, size) where
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Hiding the representation

```
module Set (Set, empty, insert, isin, size) where
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-- Implementation
data Set a = S [a]
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Hiding the representation

```
module Set (Set, empty, insert, isin, size) where
-- Interface
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-- Implementation
data Set a = S [a]

empty = S []
insert x (S xs) = S(if elem x xs then xs else x:xs)
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Cannot construct values of type `Set` outside of module `Set` because `S` is not exported

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```

Cannot construct values of type Set outside of module Set because S is not exported

```
Test.hs:3:11:  Not in scope:  data constructor ‘S’
```

Uniform naming convention: S $\rightsquigarrow$ Set

```
module Set (Set, empty, insert, isin, size) where
-- Interface
...
-- Implementation
data Set a = Set [a]

empty = Set []
insert x (Set xs) = Set(if elem x xs then xs else x:xs)
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```

Which Set is exported?

Slightly more efficient: newtype

```
module Set (Set, empty, insert, isin, size) where
-- Interface
...
-- Implementation
newtype Set a = Set [a]

empty = Set []
insert x (Set xs) = Set(if elem x xs then xs else x:xs)
isin x (Set xs) = elem x xs
size (Set xs) = length xs
```

Conceptual insight

Data representation can be hidden
by wrapping data up in a constructor that is not exported

What if Set is already a data type?

```
module SetByTree (Set, empty, insert, isin, size) where

-- Interface
empty  :: Set a
insert :: Ord a => a -> Set a -> Set a
isin   :: Ord a => a -> Set a -> Bool
size   :: Set a -> Integer

-- Implementation
type Set a = Tree a
data Tree a = Empty | Node a (Tree a) (Tree a)
```


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-- Implementation
type Set a = Tree a
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```


No need for newtype:
The representation of Tree is hidden
as long as its constructors are hidden

Beware of ==

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module SetByTree (Set, empty, insert, isin, size) where
...
type Set a = Tree a
data Tree a = Empty | Node a (Tree a) (Tree a)
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Class instances are automatically exported and cannot be hidden

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Client module:

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import SetByTree
... insert 2 (insert 1 empty) ==
    insert 1 (insert 2 empty)
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Result is probably False — representation is partly exposed!

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The proper treatment of ==

Some alternatives:

- Do not make Tree an instance of Eq
- Hide representation:

```
-- do not export constructor Set:
newtype Set a = Set (Tree a)
data Tree a = Empty | Node a (Tree a) (Tree a)
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The proper treatment of ==

Some alternatives:

- Do not make Tree an instance of Eq
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```
-- do not export constructor Set:
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data Tree a = Empty | Node a (Tree a) (Tree a)
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- Define the right == on Tree:

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For uniformity we write $\{a\}$ for the type of finite sets over type a

From lists to sets

Each list $[x_1, \dots, x_n]$ represents the set $\{x_1, \dots, x_n\}$.

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In Haskell style: α `[] = {}`

α `(x:xs) = {x}` $\cup$ α `xs`

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α `(insert x xs)` = `{x}` $\cup$ α `xs`

`isin x xs` = `x` $\in$ α `xs`

`size xs` = $|\alpha$ `xs`$|$

For the mathematically enclined:

α must be a homomorphism

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In Haskell style:

```
α []     = {}
α (x:xs) = {x} ∪ α xs
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What does it mean that "lists simulate (implement) sets":

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α (concrete operation)  =  abstract operation
α empty                 =  {}
α (insert x xs)         =  {x} ∪ α xs
isin x xs               =  x ∈ α xs
size xs                 =  |α xs|
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Implementation I: lists with duplicates

```
empty        =  []
insert x xs  =  x : xs
isin x xs    =  elem x xs
size xs      =  length(nub xs)
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The simulation requirements:

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α empty  =  {}
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$$\texttt{isin x xs} = \texttt{x} \in \alpha\ \texttt{xs}$$

$$\texttt{size xs} = |\alpha\ \texttt{xs}|$$

Two proofs immediate, two need lemmas proved by induction

Implementation II: lists without duplicates

```
empty        = []
insert x xs  = if elem x xs then xs else x:xs
isin x xs    = elem x xs
size xs      = length xs
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$$\alpha\ (\texttt{insert x xs}) = \{\texttt{x}\} \cup \alpha\ \texttt{xs}$$

From lists to sets

Each list $[x_1, \ldots, x_n]$ represents the set $\{x_1, \ldots, x_n\}$.

Abstraction function α `:: [a] -> {a}`

$\alpha[x_1, \ldots, x_n] = \{x_1, \ldots, x_n\}$

In Haskell style:

```
α []     = {}
α (x:xs) = {x} ∪ α xs
```

What does it mean that "lists simulate (implement) sets":

```
α (concrete operation)  =  abstract operation
α empty                 =  {}
α (insert x xs)         =  {x} ∪ α xs
isin x xs               =  x ∈ α xs
size xs                 =  |α xs|
```

For the mathematically enclined:
α must be a homomorphism

Implementation II: lists without duplicates

```
empty       = []
insert x xs = if elem x xs then xs else x:xs
isin x xs   = elem x xs
size xs     = length xs
```

The simulation requirements:

```
α empty          =  {}
α (insert x xs)  =  {x} ∪ α xs
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Needs *invariant* that xs contains no duplicates

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Needs *invariant* that xs contains no duplicates

```
invar :: [a] -> Bool
invar []     = True
invar (x:xs) = not(elem x xs) && invar xs
```

Implementation II: lists without duplicates

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empty        =  []
insert x xs  =  if elem x xs then xs else x:xs
isin x xs    =  elem x xs
size xs      =  length xs
```

Revised simulation requirements:

$$
\begin{array}{rrcl}
 & \alpha\ \texttt{empty} & = & \{\} \\
\texttt{invar xs} \Longrightarrow & \alpha\ (\texttt{insert x xs}) & = & \{\texttt{x}\} \cup \alpha\ \texttt{xs} \\
\texttt{invar xs} \Longrightarrow & \texttt{isin x xs} & = & \texttt{x} \in \alpha\ \texttt{xs} \\
\texttt{invar xs} \Longrightarrow & \texttt{size xs} & = & |\alpha\ \texttt{xs}|
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\end{array}
$$

Proofs omitted.

Implementation II: lists without duplicates

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Implementation II: lists without duplicates

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Revised simulation requirements:

$$\alpha\ \texttt{empty} = \{\}$$

$$\texttt{invar xs} \Longrightarrow \alpha\ \texttt{(insert x xs)} = \{\texttt{x}\} \cup \alpha\ \texttt{xs}$$

$$\texttt{invar xs} \Longrightarrow \texttt{isin x xs} = \texttt{x} \in \alpha\ \texttt{xs}$$

invar must be invariant!

In an imperative context:

If invar is true before an operation,
it must also be true after the operation

In a functional context:

If invar is true for the arguments of an operation,
it must also be true for the result of the operation

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$$\texttt{invar empty}$$

$$\texttt{invar xs} \Longrightarrow \texttt{invar (insert x xs)}$$

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If `invar` is true for the arguments of an operation,
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`invar` is *preserved* by every operation

```
               invar empty
invar xs  ==>  invar (insert x xs)
```

Proofs do not even need induction

Summary

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Let C and A be two modules that have the same interface:
a type T and a set of functions F
To prove that C is a correct implementation of A define
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an *abstraction function* α :: $C.T$ -> $A.T$
and an *invariant* `invar` :: $C.T$ -> `Bool`

Summary

Let C and A be two modules that have the same interface:
a type T and a set of functions F
To prove that C is a correct implementation of A define
an *abstraction function* α :: $C.T$ -> $A.T$
and an *invariant* invar :: $C.T$ -> Bool
and prove for each $f \in F$:

- invar is invariant:

$$\texttt{invar}\ x_1 \wedge \cdots \wedge \texttt{invar}\ x_n \implies \texttt{invar}\ (C.f\ x_1\ \ldots\ x_n)$$

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(where invar is True on types other than $C.T$)

- $C.f$ simulates $A.f$:

$$\texttt{invar}\ x_1 \wedge \cdots \wedge \texttt{invar}\ x_n \implies$$
$$\alpha(C.f\ x_1\ \ldots\ x_n) = A.f\ (\alpha\ x_1)\ \ldots\ (\alpha\ x_n)$$

(where α is the identity on types other than $C.T$)

11. Case Study: Huffman Coding