Script generated by TTT

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Finding primitive recursive definitions

Example





Primitive recursion on lists:

$$f [] = base -- base case$$

 $f (x : xs) = rec -- recursive case$

- base: no call of f
- rec: only call(s) f xs

f may have additional parameters.



Beyond primitive recursion: Multiple arguments

Example



General recursion: Quicksort

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Example

```
quicksort :: Ord a \Rightarrow [a] \rightarrow [a]
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General recursion: Quicksort

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quicksort [] = []
quicksort (x:xs) =
  quicksort below ++ [x] ++ quicksort above
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General recursion: Quicksort

Example

```
quicksort :: Ord a \Rightarrow [a] \rightarrow [a]
quicksort [] = []
quicksort (x:xs) =
  quicksort below ++ [x] ++ quicksort above
     where
       below = [y \mid y \leftarrow xs, y \leftarrow x]
       above = [y \mid y \leftarrow xs, x \leftarrow y]
```



Accumulating parameter

Idea: Result is accumulated in parameter and returned later



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-- 1st param: input list

-- 2nd param: partial ascending sublist



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Idea: Result is accumulated in parameter and returned later Example: list of all (maximal) ascending sublists in a list ups [3,0,2,3,2,4] = [[3], [0,2,3], [2,4]]

ups2 :: Ord a =>
$$[a] \rightarrow [a] \rightarrow [[a]]$$

-- 1st param: input list

-- 2nd param: partial ascending sublist (reversed)



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Convention

Mutual recursion

Example

```
even :: Int -> Bool
even n = n == 0 || n > 0 && odd (n-1) || odd (n+1)

odd :: Int -> Bool
odd n = n /= 0 && (n > 0 && even (n-1) || even (n+1))
```



Scoping by example

Identifiers of list type end in 's':

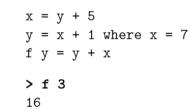
xs, ys, zs, ...



Scoping by example

$$x = y + 5$$

 $y = x + 1$ where $x = 7$
 $f y = y + x$





Scoping by example

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Binding occurrence Bound occurrence Scope of binding



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Scoping by example



Summary:

- Order of definitions is irrelevant
- Parameters and where-defs are local to each equation

5. Proofs



Aim

Aim

Guarentee functional (I/O) properties of software

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- Testing can guarantee properties for some inputs.
- Mathematical proof can guarantee properties for all inputs.



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Guarentee functional (I/O) properties of software

- Testing can guarantee properties for some inputs.
- Mathematical proof can guarantee properties for all inputs.

QuickCheck is good, proof is better

Beware of bugs in the above code; I have only proved it correct, not tried it.

Donald E. Knuth, 1977

5.1 Proving properties

What do we prove?

Equations e1 = e2



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What do we prove?

Equations
$$e1 = e2$$

How do we prove them?

By using defining equations f p = t



A first, simple example

Remember: [] ++ ys = ys

$$(x:xs)$$
 ++ ys = x : $(xs$ ++ ys)



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Proof of [1,2] ++ [] = [1] ++ [2]:



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Proof of [1,2] ++ [] = [1] ++ [2]:

1:2:[] ++ []

= 1 : (2:[] ++ []) -- by def of ++

= 1 : 2 : ([] ++ []) -- by def of ++

= 1 : 2 : [] -- by def of ++
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Observation: first used equations from left to right (ok),



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Remember: [] ++ ys = ys (x:xs) ++ ys = x : (xs ++ ys)

Proof of [1,2] ++ [] = [1] ++ [2]:

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= 1:[] ++ ([]) -- by def of ++
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Observation: first used equations from left to right (ok), then from right to left (strange!)



A more natural proof of [1,2] ++ [] = [1] ++ [2]:

```
1:2:[] ++ []
= 1 : (2:[] ++ []) -- by def of ++
= 1 : 2 : ([] ++ []) -- by def of ++
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1:[] ++ 2:[]
= 1 : ([] ++ 2:[]) -- by def of ++
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= 1 : ([] ++ 2:[]) -- by def of ++
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```

Proofs of e1 = e2 are often better presented as two reductions to some expression e:

```
e1 = ... = e
e2 = ... = e
```



Fact If an equation does not contain any variables, it can be proved by evaluating both sides separately and checking that the result is identical.



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But how to prove equations with variables, for example *associativity* of ++:

$$(xs ++ ys) ++ zs = xs ++ (ys ++ zs)$$



Properties of recursive functions are proved by induction



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Induction on natural numbers: see Diskrete Strukturen

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Induction on natural numbers: see Diskrete Strukturen

Induction on lists: here and now



Structural induction on lists

To prove property P(xs) for all finite lists xs



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\uparrow

induction

hypothesis
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This is called *structural induction* on xs.



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Base case: Prove P([]) and
Induction step: Prove P(xs) implies P(x:xs)

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This is called *structural induction* on xs. It is a special case of induction on the length of xs.



Example: associativity of ++

Lemma app_assoc: (xs ++ ys) ++ zs = xs ++ (ys ++ zs)



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To show: ((x:xs) ++ ys) ++ zs = (x:xs) ++ (ys ++ zs)



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To show: ([] ++ ys) ++ zs = [] ++ (ys ++ zs)
 ([] ++ ys) ++ zs
 = VS ++ ZS
                   -- by def of ++
 = [] ++ (ys ++ zs) -- by def of ++
Induction step:
To show: ((x:xs) ++ ys) ++ zs = (x:xs) ++ (ys ++ zs)
 ((x:xs) ++ ys) ++ zs
= (x : (xs ++ ys)) ++ zs -- by def of ++
 = x : ((xs ++ ys) ++ zs) -- by def of ++
```

= x : (xs ++ (ys ++ zs)) -- by IH



Example: associativity of ++

Induction step:

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(x:xs) ++ (ys ++ zs)
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Lemma app_assoc: (xs ++ ys) ++ zs = xs ++ (ys ++ zs)
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 = x : (xs ++ (ys ++ zs)) -- by IH
 (x:xs) ++ (ys ++ zs)
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Induction template

Lemma P(xs)



Induction template

Lemma P(xs)

Proof by structural induction on xs

Base case:

To show: P([])



Induction template



Induction template

Lemma P(xs)

Proof by structural induction on xs

Base case:

To show: P([])

Proof of P([])

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Lemma P(xs)
```

Proof by structural induction on xs

Base case:

To show: P([])

Proof of P([])

Induction step:

To show: P(x:xs)

Proof of P(x:xs) using IH P(xs)



Example: length of ++

Lemma length(xs ++ ys) = length xs + length ys

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Lemma length(xs ++ ys) = length xs + length ys

Proof by structural induction on xs

Base case:

```
To show: length ([] ++ ys) = length [] + length ys
length ([] ++ ys)
 = length ys
                        -- by def of ++
length [] + length vs
 = 0 + length vs
                        -- by def of length
 = length ys
```



Example: length of ++

Lemma length(xs ++ ys) = length xs + length ys

Proof by structural induction on xs

Base case:

To show: length ([] ++ ys) = length [] + length ys length ([] ++ ys) = length ys -- by def of ++ length [] + length ys = 0 + length ys -- by def of length = length ys Induction step: To show: length((x:xs)++ys) = length(x:xs) + length ys



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Lemma length(xs ++ ys) = length xs + length ys

Proof by structural induction on xs

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Induction step:
To show: length((x:xs)++ys) = length(x:xs) + length ys
length((x:xs) ++ ys)
 = length(x : (xs ++ ys)) -- by def of ++
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Example: length of ++

-- by def of length

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Example: reverse of ++

Lemma reverse(xs ++ ys) = reverse ys ++ reverse xs



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= reverse ys -- by def of ++
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= reverse ys -- by Lemma app_Nil2
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To show: reverse((x:xs)++ys) = reverse ys ++ reverse(x:xs)



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Induction step:



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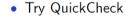
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