Script generated by TTT

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`HOL/Data_Structures/RBT_Set.thy`

Relationship to 2-3-4 trees

Idea: encode 2-3-4 trees as binary trees;

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use color to express grouping

$$\langle\rangle \approx \langle\rangle$$

Color

Structural invariants

$invh :: {'a}\ rbt \Rightarrow bool$

$invh\ \langle\rangle = True$

$invh\ \langle\_,\ l,\ \_,\ r\rangle = (invh\ l \wedge invh\ r \wedge bh(l) = bh(r))$

Logarithmic height

Lemma

$rbt\ t \Longrightarrow h(t) \leq 2 * \log_2 |t|_1$

Insertion

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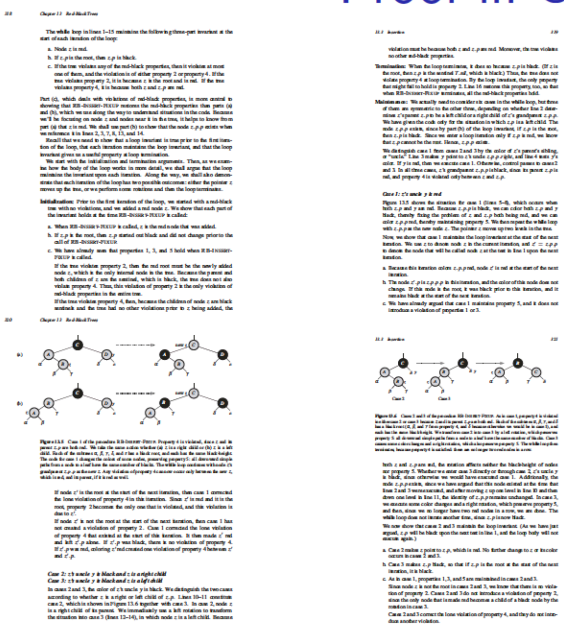
Proof in CLRS



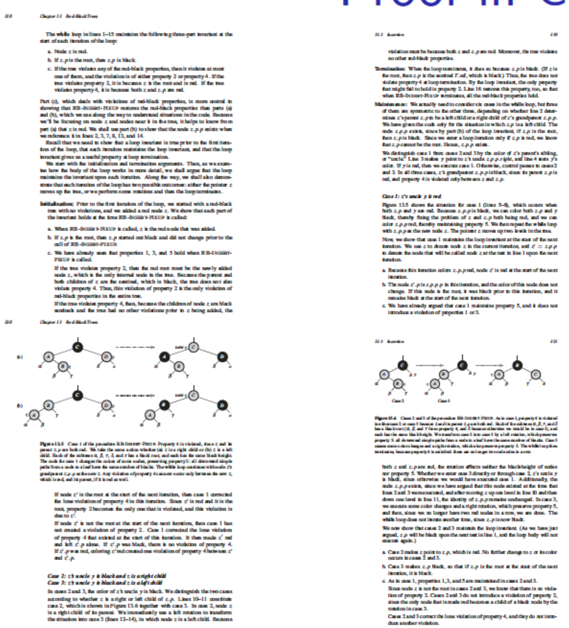
Preservation of invariant

After 14 simple lemmas:

Theorem

$rbt\ t \Longrightarrow rbt\ (insert\ x\ t)$

Proof in CLRS



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$\mid EQ \Rightarrow combine\ l\ r$
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Deletion

Tricky functions: $baldL,\ baldR,\ combine$

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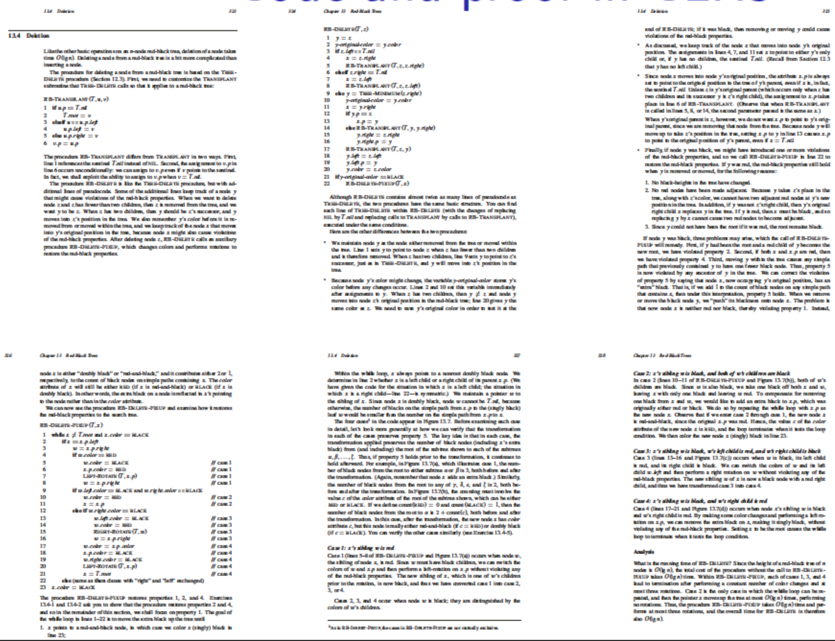
12 short but tricky to find invariant lemmas with short proofs. The worst:

$[\![invh\ t;\ invc\ t]\!]$
$\Longrightarrow invh\ (del\ x\ t) \wedge$
$\quad (color\ t = Red \wedge$
$\quad\ bh(del\ x\ t) = bh(t) \wedge invc\ (del\ x\ t) \vee$
$\quad\ color\ t = Black \wedge$
$\quad\ bh(del\ x\ t) = bh(t) - 1 \wedge invc2\ (del\ x\ t))$

Code and proof in CLRS



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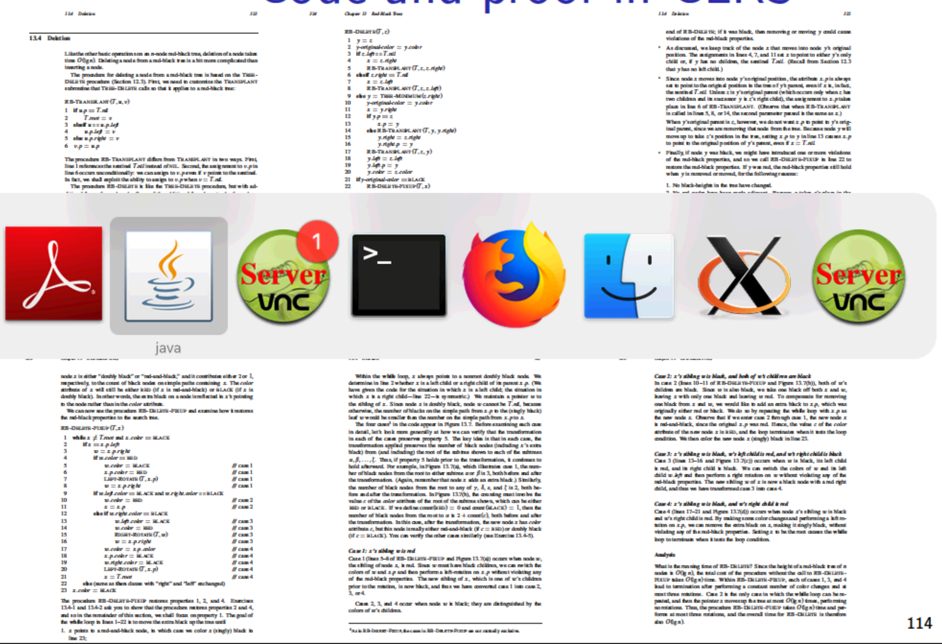

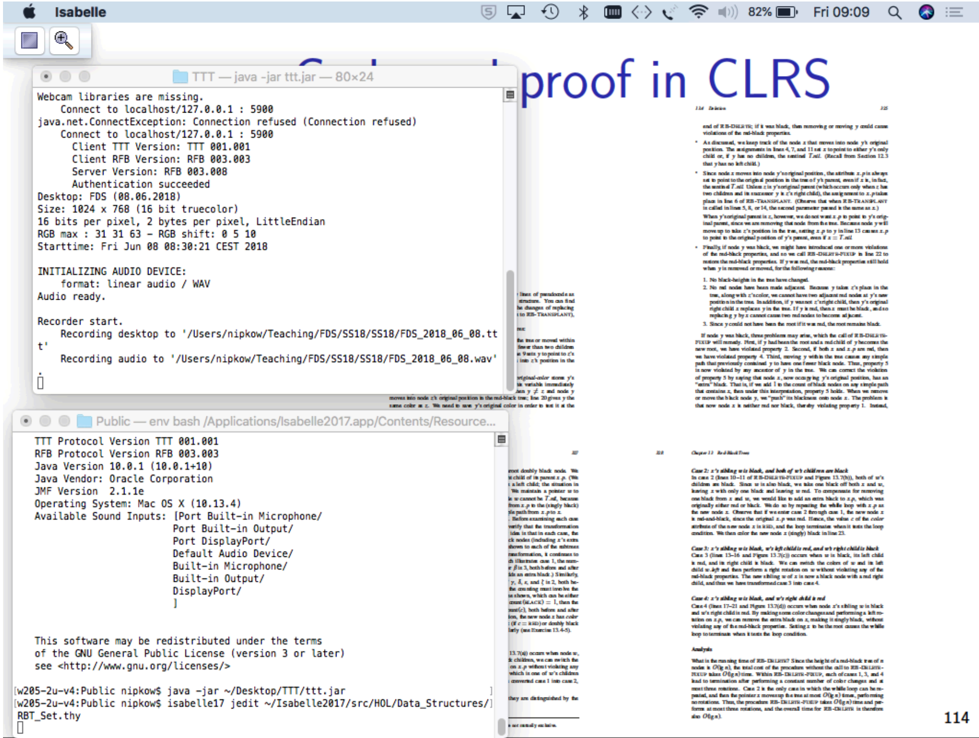

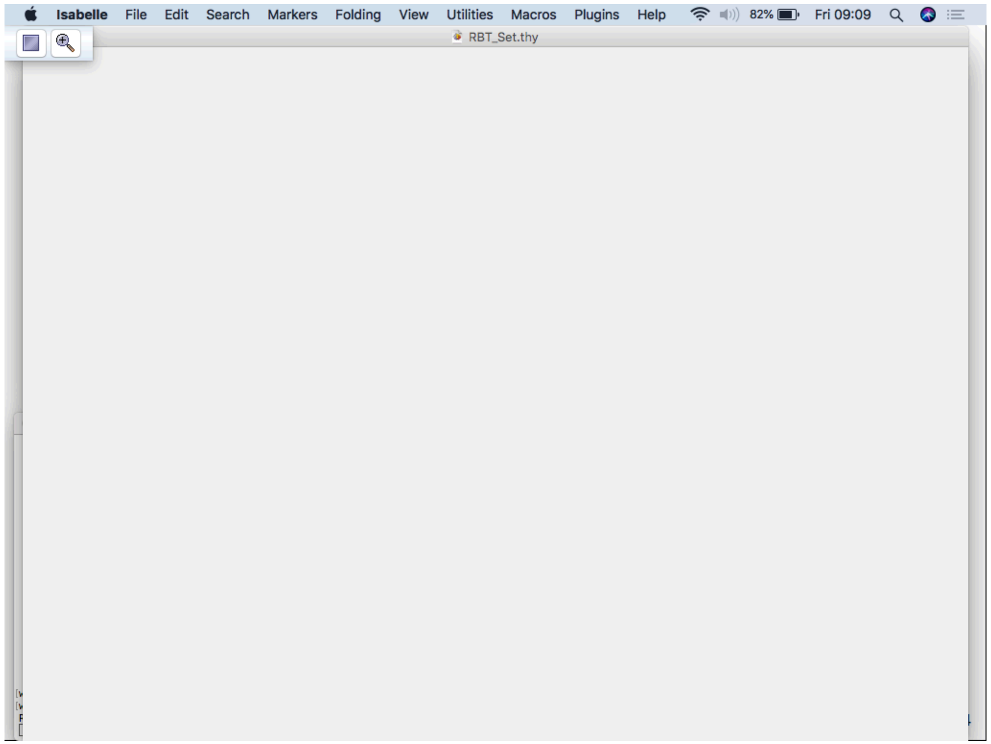

Source of code

Insertion:

Okasaki's *Purely Functional Data Structures*

Deletion:

Stefan Kahrs. *Red Black Trees with Types.*
J. Functional Programming. 1996.

AVL Trees

[Adelson-Velskii & Landis 62]

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- Every node $\langle l, \_, r \rangle$ must be balanced:
  $|h(l) - h(r)| \leq 1$

Weight-Balanced Trees

[Nievergelt & Reingold 72,73]

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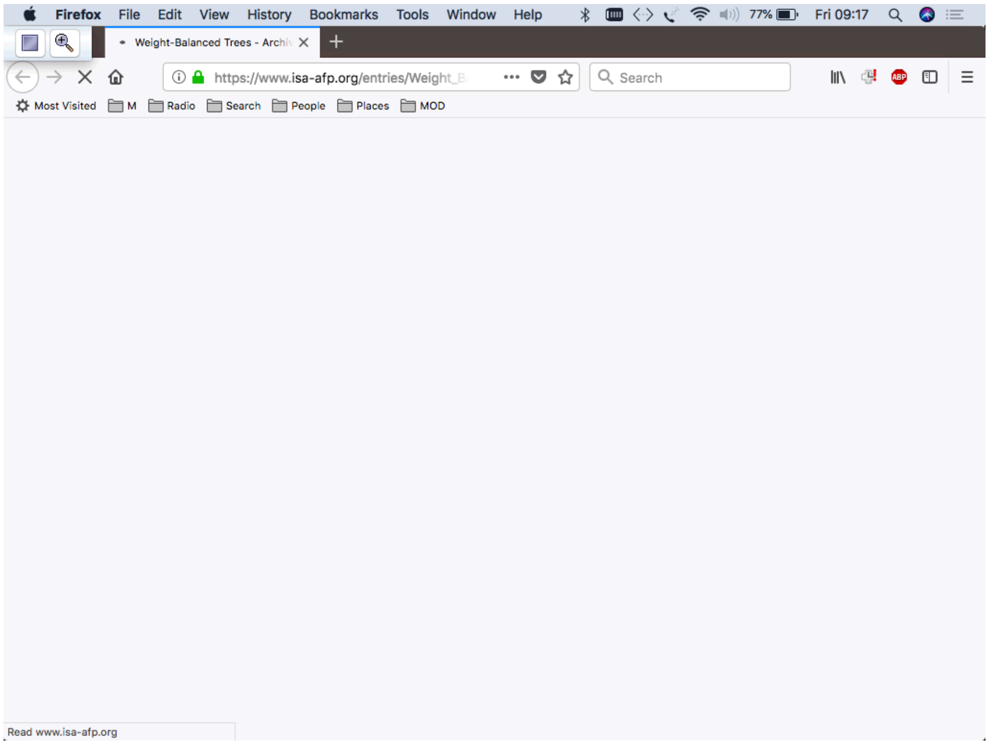
- Verified implementation in Isabelle's Archive of Formal Proofs.



AA trees

[Arne Andersson 93, Ragde 14]

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- Simulation of 2-3 trees by binary trees
  $\langle t_1, a, t_2, b, t_3 \rangle \;\leadsto\; \langle t_1, a, \langle t_2, b, t_3 \rangle \rangle$

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- 4 bugs in *delete* in [Ragde 14]:
  non-linear pattern; going down wrong subtree; missing function call; off by 1

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After corrections, the proofs:

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AA trees

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After corrections, the proofs:

- Code relies on tricky pre- and post-conditions that need to be found
- Structural invariant preservation requires most of the work

12 More Search Trees

Scapegoat trees

[Anderson 89, Igal & Rivest 93]

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Central idea:

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Rebuild when the tree gets "too unbalanced"

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- Tricky: amortized logarithmic complexity analysis

One by one (Union)

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Adding smaller set to bigger or bigger to smaller?

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- Bigger into smaller:
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- $c(x) = \log_2 x \implies$
  Cost = $O(m * \log_2(n+m))$

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- Bigger into smaller:
  Cost $= c(m) + \cdots + c(m+n-1)$
- $c(x) = \log_2 x \Longrightarrow$
  Cost $= O(m * \log_2(n+m)) = O(m * \log_2 n)$

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- This section:

  A parallel divide and conquer approach
- Cost: $O(m * \log_2(\frac{n}{m} + 1))$
- Works for many kinds of balanced trees
- For ease of presentation: use concrete type *tree*

Uniform *tree* type

Red-Black trees, AVL trees, weight-balanced trees, etc can all be implemented with one more field per node:

datatype $('a,\ 'b)\ \mathit{tree} = \langle\rangle$
$\quad |\ \mathit{Node}\ 'b\ (('a,\ 'b)\ \mathit{tree})\ 'a\ (('a,\ 'b)\ \mathit{tree})$

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We work with this type of trees without committing to any particular kind of balancing schema.

Syntax:

$$\langle b, l, a, r\rangle \equiv Node\ b\ l\ a\ r$$

Just *join*

Can synthesize all BST interface functions from just one function:

$join\ l\ a\ r$

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$join\ l\ a\ r \approx Node\ \_\ l\ a\ r +$ rebalance

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$union :: tree \Rightarrow tree \Rightarrow tree$

13 Union, Intersection, Difference on BSTs

Correctness

join for Red-Black Trees

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Given $join :: tree \Rightarrow 'a \Rightarrow tree \Rightarrow tree$
(where *tree* abbreviates $('a,'b)\ tree$), implement

$split :: tree \Rightarrow 'a \Rightarrow tree \times bool \times tree$
$insert :: 'a \Rightarrow tree \Rightarrow tree$
$union :: tree \Rightarrow tree \Rightarrow tree$
$join2 :: tree \Rightarrow tree \Rightarrow tree$
$delete :: 'a \Rightarrow tree \Rightarrow tree$