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Chapter 4

Logic and Proof Beyond Equality

maximize

Syntax (in decreasing precedence):

$$
\begin{array}{lllllll}
form & ::= & (form) & \mid & term = term & \mid & \neg form \\
 & \mid & form \wedge form & \mid & form \vee form & \mid & form \longrightarrow form \\
 & \mid & \forall x.\ form & \mid & \exists x.\ form & &
\end{array}
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Input syntax: $\longleftrightarrow$ (same precedence as $\longrightarrow$)

Variable binding convention:

$$\forall\, x\ y.\ P\ x\ y \ \equiv\ \forall\, x.\ \forall\, y.\ P\ x\ y$$

Warning

Quantifiers have low precedence
and need to be parenthesized (if in some context)

! $P \wedge \forall\, x.\ Q\ x \ \leadsto\ P \wedge (\forall\, x.\ Q\ x)$ **!**

Mathematical symbols

. . . and their ascii representations:

$\forall$	`\<forall>`	`ALL`
$\exists$	`\<exists>`	`EX`
λ	`\<lambda>`	`%`
$\longrightarrow$	`-->`	
$\longleftrightarrow$	`<->`	
$\wedge$	`/\`	`&`
$\vee$	`\/`	`|`
$\neg$	`\<not>`	`~`
$\neq$	`\<noteq>`	`~=`

Sets over type $'a$

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$\in$	`\<in>`	`:`
$\subseteq$	`\<subseteq>`	`<=`
$\cup$	`\<union>`	`Un`
$\cap$	`\<inter>`	`Int`

5 Logical Formulas

6 Proof Automation

7 Single Step Proofs

simp and *auto*

simp: rewriting and a bit of arithmetic

auto: rewriting and a bit of arithmetic, logic and sets

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- Show you where they got stuck

simp and *auto*

simp: rewriting and a bit of arithmetic

auto: rewriting and a bit of arithmetic, logic and sets

- Show you where they got stuck
- highly incomplete
- Extensible with new *simp*-rules

Exception: *auto* acts on all subgoals

fastforce

- rewriting, logic, sets, relations and a bit of arithmetic.

fastforce

- rewriting, logic, sets, relations and a bit of arithmetic.
- incomplete but better than *auto*.
- Succeeds or fails

blast

- A complete proof search procedure for FOL . . .

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- . . . but (almost) without "="

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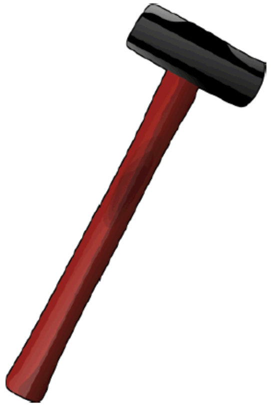
- A complete proof search procedure for FOL . . .
- . . . but (almost) without "="
- Covers logic, sets and relations

Sledgehammer



chitecture:

Isabelle

external
ATPs[1]

[1] Automatic Theorem Provers

chitecture:

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Goal
& filtered library ↓

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[1] Automatic Theorem Provers

Architecture:

Isabelle

$$\begin{array}{ccc} \text{Goal} & & \\ \text{\& filtered library} & \downarrow \;\; \uparrow & \text{Proof} \end{array}$$

external
ATPs[1]

Characteristics:

- Sometimes it works,
- sometimes it doesn't.

[1] Automatic Theorem Provers

by$(proof\text{-}method)$

$\approx$

apply$(proof\text{-}method)$
done

`Auto_Proof_Demo.thy`

Linear formulas

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Only:

variables

numbers

number $*$ variable

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$+, -$

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$=, \leq, <$

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Examples

Linear: $3 * x + 5 * y \leq z \longrightarrow x < z$

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Examples

Linear: $3 * x + 5 * y \leq z \longrightarrow x < z$

Nonlinear: $x \leq x * x$

Extended linear formulas

Also allowed:

min, max

$even$, odd

$t\ div\ n$, $t\ mod\ n$ where n is a number

conversion functions

nat, $floor$, $ceiling$, abs

Automatic proof of arithmetic formulas

by *arith*

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Proof method *arith* tries to prove arithmetic formulas.

- Succeeds or fails
- Decision procedure for extended linear formulas

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by *arith*

Proof method *arith* tries to prove arithmetic formulas.

- Succeeds or fails
- Decision procedure for extended linear formulas
- Nonlinear subterms are viewed as (new) variables.
  Example: $x \leq x * x + f\ y$ is viewed as $x \leq u + v$

Automatic proof of arithmetic formulas

by (*simp add: algebra_simps*)

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- It contains associativity, commutativity and distributivity of $+$ and $*$.
- This may prove the formula, may make it simpler, or may make it unreadable.

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Example

simp add: numeral_eq_Suc rewrites $x \,\hat{}\, 2$ to $x * x$

`Auto_Proof_Demo.thy`

Arithmetic

Step-by-step proofs can be necessary if automation fails and you have to explore where and why it failed by taking the goal apart.

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"Backchaining"

Typical backwards rules

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They are known as introduction rules
because they *introduce* a particular connective.

Forward proof: OF

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```
conjI[OF refl[of "a"]]
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From now on: $?$ mostly suppressed on slides

Single_Step_Demo.thy

$\Longrightarrow$ versus $\longrightarrow$

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Bookmarks

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